

# THE METRIZABILITY OF THE GENERALIZED TANGENT BUNDLE OF A VECTOR BUNDLE

by  
CONSTANTIN M. ARCUS

## Abstract

A class of metrizable vector bundles have been presented in the paper [5]. Using a generalized Lie algebroid we obtain the Lie algebroid generalized tangent bundle. This Lie algebroid is a new example of metrizable vector bundle. A new class of Lagrange spaces, called by use, generalized Lagrange  $(\rho, \eta)$ -space, Lagrange  $(\rho, \eta)$ -space and Finsler  $(\rho, \eta)$ -space are presented. The results obtained in the particular case of Lie algebroids emphasize the importance and the utility of our new method by work. In particular, if all morphisms are identities morphisms, then similar results with classical results are obtained.

**2000 Mathematics Subject Classification:** 53C05, 53C07, 53C60, 58B20.

**Keywords:** vector bundle, (generalized) Lie algebroid, (linear) connection, natural base, adapted base, (pseudo)metrical structure, distinguished linear connection, metrizable vector bundle.

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# 1 Introduction

The study of the geometry of the usual Lie algebroid

$$((TTM, \tau_{TM}, TM), [\cdot]_{TTM}, (Id_{TTM}, Id_{TM}))$$

with a metrical structure

$$g = g_{ij} dy^i \otimes dy^j \in \mathcal{T}_2^0(VTTM, \tau_{TM}, TM),$$

was extensively examined by geometers and physicists in the framework of generalized Lagrange space. The generalized Lagrange spaces were introduced and studied by R. Miron [19]. See also [2, 3, 4] for this topic.

We know that a regular Lagrangian on  $TM$  is a smooth function  $TM \xrightarrow{L} \mathbb{R}$  such that the Hessian matrix with entries

$$g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 L(x, y)}{\partial y^i \partial y^j}$$

is everywhere nondegenerate. If the metrical structure of a generalized Lagrange space is determined by a regular Lagrangian, then we obtain the Lagrange space. (see [21]). The notion of Lagrange space was introduced and studied by J. Kern [15] and R. Miron [17, 18]. The nonlinear connections and the distinguished linear connections which depend only on Lagrangian were presented in the framework of Lagrange space. The geometry of Lagrange spaces have been developed in many proceedings and monographs. [7, 9, 11, 20, 21, 22, 23].

The case when  $L$  is square of a function on  $TM$ , positively, 1-homogeneous with respect to the velocity  $y^i$ , provides an important class of Lagrange spaces called Finsler spaces. The geometry of Finsler space is a subgeometry of the geometry of the Lie algebroid

$$((TTM, \tau_{TM}, TM), [\cdot]_{TTM}, (Id_{TTM}, Id_{TM})).$$

Important contributions to the geometry of Finsler spaces were obtained by M. Abate and G. Patritio [1], D. Bao, S. S. Cern and Z. Shen [8], A. Bejancu [9], L. Berwald [10], H. Busmann [12], E. Cartan [13].

The classical results, were extended to the study of the geometry of the usual Lie algebroid  $((TE, \tau_E, E), [\cdot]_{TE}, (Id_{TE}, Id_E))$ , where  $(E, \pi, M)$  is a vector bundle. (see [19, 24, 25])

The generalized Lie algebroid (see [5]) is a new notion necessary to obtain a new class of nonlinear connections in Ehresmann sense. Using that, we obtain the Lie algebroid generalized tangent bundle  $\left( ((\rho, \eta)TE, (\rho, \eta)\tau_E, E), [\cdot]_{(\rho, \eta)TE}, (\tilde{\rho}, Id_E) \right)$ . In the Sections 2, 3 and 4 we present the basic notions and terminology. The study of the metrizable of this Lie algebroid is our objective in Section 5. In the particular case of Lie algebroids, we obtain important results. Finally, in Section 6, we introduced a new class of Lagrange space, called by use *the generalized Lagrange  $(\rho, \eta)$ -space, Lagrange  $(\rho, \eta)$ -space and Finsler  $(\rho, \eta)$ -space*.

New and important results are obtained in the particular case of Lie algebroids. In particular, if  $(\rho, \eta, h) = (Id_{TM}, Id_M, Id_M)$ , then similar results with the classical results are obtained.

## 2 Preliminaries

Let **Vect**, **Liealg**, **Mod**, **Man** and  $\mathbf{B}^{\mathbf{v}}$  be the category of real vector spaces, Lie algebras, modules, manifolds and vector bundles respectively.

We know that if  $(E, \pi, M) \in |\mathbf{B}^{\mathbf{v}}|$ ,  $\Gamma(E, \pi, M) = \{u \in \mathbf{Man}(M, E) : u \circ \pi = Id_M\}$  and  $\mathcal{F}(M) = \mathbf{Man}(M, \mathbb{R})$ , then  $(\Gamma(E, \pi, M), +, \cdot)$  is a  $\mathcal{F}(M)$ -module. If  $(\varphi, \varphi_0) \in \mathbf{B}^{\mathbf{v}}((E, \pi, M), (E', \pi', M'))$  such that  $\varphi_0 \in Iso_{\mathbf{Man}}(M, M')$ , then, using the operation

$$\begin{array}{ccc} \mathcal{F}(M) \times \Gamma(E', \pi', M') & \xrightarrow{\quad \cdot \quad} & \Gamma(E', \pi', M') \\ (f, u') & \longmapsto & f \circ \varphi_0^{-1} \cdot u' \end{array}$$

it results that  $(\Gamma(E', \pi', M'), +, \cdot)$  is a  $\mathcal{F}(M)$ -module and we obtain the **Mod**-morphism

$$\begin{array}{ccc} \Gamma(E, \pi, M) & \xrightarrow{\Gamma(\varphi, \varphi_0)} & \Gamma(E', \pi', M') \\ u & \longmapsto & \Gamma(\varphi, \varphi_0)u \end{array}$$

defined by

$$\Gamma(\varphi, \varphi_0)u(y) = \varphi\left(u_{\varphi_0^{-1}(y)}\right),$$

for any  $y \in M'$ .

Let  $M, N \in |\mathbf{Man}|$ ,  $h \in Iso_{\mathbf{Man}}(M, N)$  and  $\eta \in Iso_{\mathbf{Man}}(N, M)$ .

We know (see [5]) that if  $(F, \nu, N) \in |\mathbf{B}^{\mathbf{v}}|$  so that there exists

$$(\rho, \eta) \in \mathbf{B}^{\mathbf{v}}((F, \nu, N), (TM, \tau_M, M))$$

and also an operation

$$\begin{array}{ccc} \Gamma(F, \nu, N) \times \Gamma(F, \nu, N) & \xrightarrow{[\cdot]_{F,h}} & \Gamma(F, \nu, N) \\ (u, v) & \longmapsto & [u, v]_{F,h} \end{array}$$

with the following properties:

*GLA*<sub>1</sub>. the equality holds good

$$[u, f \cdot v]_{F,h} = f[u, v]_{F,h} + \Gamma(Th \circ \rho, h \circ \eta)(u) f \cdot v,$$

for all  $u, v \in \Gamma(F, \nu, N)$  and  $f \in \mathcal{F}(N)$ .

*GLA*<sub>2</sub>. the 4-tuple  $(\Gamma(F, \nu, N), +, \cdot, [\cdot]_{F,h})$  is a Lie  $\mathcal{F}(N)$ -algebra,

*GLA*<sub>3</sub>. the **Mod**-morphism  $\Gamma(Th \circ \rho, h \circ \eta)$  is a **LieAlg**-morphism of

$$\left(\Gamma(F, \nu, N), +, \cdot, [\cdot]_{F,h}\right)$$

source and

$$(\Gamma(TN, \tau_N, N), +, \cdot, [\cdot]_{TN})$$

target, then the triple  $((F, \nu, N), [\cdot]_{F,h}, (\rho, \eta))$  is called *generalized Lie algebroid*.

In particular, if  $h = Id_M = \eta$ , then we obtain the definition of the Lie algebroid.

Let  $((F, \nu, N), [\cdot]_{F,h}, (\rho, \eta))$  be an generalized Lie algebroid.

- Locally, for any  $\alpha, \beta \in \overline{1, p}$ , we set  $[t_\alpha, t_\beta]_{F, h} = L_{\alpha\beta}^\gamma t_\gamma$ . We easily obtain that  $L_{\alpha\beta}^\gamma = -L_{\beta\alpha}^\gamma$ , for any  $\alpha, \beta, \gamma \in \overline{1, p}$ .

The real local functions  $L_{\alpha\beta}^\gamma$ ,  $\alpha, \beta, \gamma \in \overline{1, p}$  will be called the *structure functions of the generalized Lie algebroid*  $\left((F, \nu, N), [\cdot, \cdot]_{F, h}, (\rho, \eta)\right)$ .

- We assume the following diagrams:

$$\begin{array}{ccccc}
F & \xrightarrow{\rho} & TM & \xrightarrow{Th} & TN \\
\downarrow \nu & & \downarrow \tau_M & & \downarrow \tau_N \\
N & \xrightarrow{\eta} & M & \xrightarrow{h} & N \\
(\chi^{\tilde{i}}, z^\alpha) & & (x^i, y^i) & & (\chi^{\tilde{i}}, z^{\tilde{i}})
\end{array}$$

where  $i, \tilde{i} \in \overline{1, m}$  and  $\alpha \in \overline{1, p}$ .

If

$$\begin{aligned}
(\chi^{\tilde{i}}, z^\alpha) &\longrightarrow (\chi^{\tilde{i}'}(\chi^{\tilde{i}}), z^{\alpha'}(\chi^{\tilde{i}}, z^\alpha)), \\
(x^i, y^i) &\longrightarrow (x^{\tilde{i}'}(x^i), y^{\tilde{i}'}(x^i, y^i))
\end{aligned}$$

and

$$(\chi^{\tilde{i}}, z^{\tilde{i}}) \longrightarrow (\chi^{\tilde{i}'}(\chi^{\tilde{i}}), z^{\tilde{i}'}(\chi^{\tilde{i}}, z^{\tilde{i}})),$$

then

$$z^{\alpha'} = \Lambda_\alpha^{\alpha'} z^\alpha,$$

$$y^{\tilde{i}'} = \frac{\partial x^{\tilde{i}'}}{\partial x^i} y^i$$

and

$$z^{\tilde{i}'} = \frac{\partial \chi^{\tilde{i}'}}{\partial \chi^{\tilde{i}}} z^{\tilde{i}}.$$

- We assume that  $(\theta, \mu) \stackrel{put}{=} (Th \circ \rho, h \circ \eta)$ . If  $z^\alpha t_\alpha \in \Gamma(F, \nu, N)$  is arbitrary, then

$$\begin{aligned}
(2.1) \quad & \Gamma(Th \circ \rho, h \circ \eta)(z^\alpha t_\alpha) f(h \circ \eta(\varkappa)) = \\
& = \left( \theta_\alpha^{\tilde{i}} z^\alpha \frac{\partial f}{\partial \varkappa^{\tilde{i}}} \right) (h \circ \eta(\varkappa)) = \left( (\rho_\alpha^i \circ h)(z^\alpha \circ h) \frac{\partial f \circ h}{\partial x^i} \right) (\eta(\varkappa)),
\end{aligned}$$

for any  $f \in \mathcal{F}(N)$  and  $\varkappa \in N$ .

The coefficients  $\rho_\alpha^i$  respectively  $\theta_\alpha^{\tilde{i}}$  change to  $\rho_{\alpha'}^{\tilde{i}'}$  respectively  $\theta_{\alpha'}^{\tilde{i}'}$  according to the rule:

$$(2.2) \quad \rho_{\alpha'}^{\tilde{i}'} = \Lambda_\alpha^{\alpha'} \rho_\alpha^i \frac{\partial x^{\tilde{i}'}}{\partial x^i},$$

respectively

$$(2.3) \quad \theta_{\alpha'}^{\tilde{i}'} = \Lambda_\alpha^{\alpha'} \theta_\alpha^{\tilde{i}} \frac{\partial \varkappa^{\tilde{i}'}}{\partial \varkappa^{\tilde{i}}},$$

where

$$\|\Lambda_{\alpha'}^{\alpha}\| = \|\Lambda_\alpha^{\alpha'}\|^{-1}.$$

*Remark 2.1* The following equalities hold good:

$$(2.4) \quad \rho_\alpha^i \circ h \frac{\partial f \circ h}{\partial x^i} = \left( \theta_\alpha^i \frac{\partial f}{\partial x^i} \right) \circ h, \forall f \in \mathcal{F}(N).$$

and

$$(2.5) \quad \left( L_{\alpha\beta}^\gamma \circ h \right) \left( \rho_\gamma^k \circ h \right) = (\rho_\alpha^i \circ h) \frac{\partial (\rho_\beta^k \circ h)}{\partial x^i} - (\rho_\beta^j \circ h) \frac{\partial (\rho_\alpha^k \circ h)}{\partial x^j}.$$

We have the  $\mathbf{B}^\vee$ -morphism

$$(2.6) \quad \begin{array}{ccc} \pi^*(h^*F) & \hookrightarrow & F \\ \pi^*(h^*\nu) \downarrow & & \downarrow \nu \\ E & \xrightarrow{h \circ \pi} & N \end{array}$$

Let  $\left( \begin{smallmatrix} \pi^*(h^*F) \\ \rho \end{smallmatrix}, Id_E \right)$  be the  $\mathbf{B}^\vee$ -morphism of  $(\pi^*(h^*F), \pi^*(h^*\nu), E)$  source and  $(TE, \tau_E, E)$  target, where

$$(2.7) \quad \begin{array}{ccc} \pi^*(h^*F) & \xrightarrow{\begin{smallmatrix} \pi^*(h^*F) \\ \rho \end{smallmatrix}} & TE \\ Z^\alpha T_\alpha(u_x) & \longmapsto & (Z^\alpha \cdot \rho_\alpha^i \circ h \circ \pi) \frac{\partial}{\partial x^i}(u_x) \end{array}$$

Using the operation

$$\Gamma(\pi^*(h^*F), \pi^*(h^*\nu), E)^2 \xrightarrow{[\cdot]_{\pi^*(h^*F)}} \Gamma(\pi^*(h^*F), \pi^*(h^*\nu), E)$$

defined by

$$(2.8) \quad \begin{aligned} [T_\alpha, T_\beta]_{\pi^*(h^*F)} &= (L_{\alpha\beta}^\gamma \circ h \circ \pi) T_\gamma, \\ [T_\alpha, fT_\beta]_{\pi^*(h^*F)} &= f (L_{\alpha\beta}^\gamma \circ h \circ \pi) T_\gamma + (\rho_\alpha^i \circ h \circ \pi) \frac{\partial f}{\partial x^i} T_\beta, \\ [fT_\alpha, T_\beta]_{\pi^*(h^*F)} &= -[T_\beta, fT_\alpha]_{\pi^*(h^*F)}, \end{aligned}$$

for any  $f \in \mathcal{F}(E)$ , it results that

$$\left( (\pi^*(h^*F), \pi^*(h^*\nu), E), [\cdot, \cdot]_{\pi^*(h^*F)}, \left( \begin{smallmatrix} \pi^*(h^*F) \\ \rho \end{smallmatrix}, Id_E \right) \right)$$

is a Lie algebroid.

### 3 Natural and adapted basis

We consider the following diagram:

$$(3.1) \quad \begin{array}{ccc} E & & (F, [\cdot, \cdot]_{F,h}, (\rho, \eta)) \\ \pi \downarrow & & \downarrow \nu \\ M & \xrightarrow{h} & N \end{array}$$

where  $(E, \pi, M)$  is a vector bundle and  $\left((F, \nu, N), [\cdot, \cdot]_{F,h}, (\rho, \eta)\right)$  is a generalized Lie algebroid.

We take  $(x^i, y^a)$  as canonical local coordinates on  $(E, \pi, M)$ , where  $i \in \overline{1, m}$  and  $a \in \overline{1, r}$ . Let

$$(x^i, y^a) \longrightarrow (x^{\check{i}}(x^i), y^{a'}(x^i, y^a))$$

be a change of coordinates on  $(E, \pi, M)$ . Then the coordinates  $y^a$  change to  $y^{a'}$  by the rule:

$$(3.2) \quad y^{a'} = M_a^{a'} y^a.$$

Let

$$(3.3) \quad \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^a}\right) \stackrel{put}{=} (\partial_i, \dot{\partial}_a)$$

be the natural base of the Lie algebroid  $((TE, \tau_E, E), [\cdot, \cdot]_{TE}, (Id_{TE}, Id_E))$ .

For any sections

$$Z^\alpha T_\alpha \in \Gamma(\pi^*(h^*F), \pi^*(h^*F), E)$$

and

$$Y^a \dot{\partial}_a \in \Gamma(VTE, \tau_E, E)$$

we obtain the section

$$\begin{aligned} Z^\alpha \tilde{\partial}_\alpha + Y^a \dot{\partial}_a &=: Z^\alpha (T_\alpha \oplus (\rho_\alpha^i \circ h \circ \pi) \partial_i) + Y^a (0_{\pi^*(h^*F)} \oplus \dot{\partial}_a) \\ &= Z^\alpha T_\alpha \oplus (Z^\alpha (\rho_\alpha^i \circ h \circ \pi) \partial_i + Y^a \dot{\partial}_a) \in \Gamma(\pi^*(h^*F) \oplus TE, \overset{\oplus}{\pi}, E). \end{aligned}$$

Since we have

$$\begin{aligned} Z^\alpha \tilde{\partial}_\alpha + Y^a \dot{\partial}_a &= 0 \\ \Updownarrow \\ Z^\alpha T_\alpha &= 0 \wedge Z^\alpha (\rho_\alpha^i \circ h \circ \pi) \partial_i + Y^a \dot{\partial}_a = 0, \end{aligned}$$

it implies  $Z^\alpha = 0$ ,  $\alpha \in \overline{1, p}$  and  $Y^a = 0$ ,  $a \in \overline{1, r}$ .

Therefore, the sections  $\tilde{\partial}_1, \dots, \tilde{\partial}_p, \dot{\partial}_1, \dots, \dot{\partial}_r$  are linearly independent.

We consider the vector subbundle  $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$  of the vector bundle  $(\pi^*(h^*F) \oplus TE, \overset{\oplus}{\pi}, E)$ , for which the  $\mathcal{F}(E)$ -module of sections is the  $\mathcal{F}(E)$ -submodule of  $(\Gamma(\pi^*(h^*F) \oplus TE, \overset{\oplus}{\pi}, E), +, \cdot)$ , generated by the set of sections  $(\tilde{\partial}_\alpha, \dot{\partial}_a)$  which is called the *natural*  $(\rho, \eta)$ -base.

The matrix of coordinate transformation on  $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$  at a change of fibred charts is

$$(3.4) \quad \left\| \begin{array}{cc} \Lambda_\alpha^{a'} \circ h \circ \pi & 0 \\ (\rho_\alpha^i \circ h \circ \pi) \frac{\partial M_b^{a'} \circ \pi}{\partial x_i} y^b & M_a^{a'} \circ \pi \end{array} \right\|.$$

We have the following

**Theorem 3.1** Let  $(\tilde{\rho}, Id_E)$  be the  $\mathbf{B}^v$ -morphism of  $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$  source and  $(TE, \tau_E, E)$  target, where

$$(3.5) \quad \begin{aligned} & (\rho, \eta) TE \xrightarrow{\tilde{\rho}} TE \\ & \left( Z^\alpha \tilde{\partial}_\alpha + Y^a \dot{\tilde{\partial}}_a \right) (u_x) \mapsto \left( Z^\alpha (\rho_\alpha^i \circ h \circ \pi) \partial_i + Y^a \dot{\partial}_a \right) (u_x) \end{aligned}$$

Using the operation

$$\Gamma((\rho, \eta) TE, (\rho, \eta) \tau_E, E)^2 \xrightarrow{[\cdot]_{(\rho, \eta) TE}} \Gamma((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$$

defined by

$$(3.6) \quad \begin{aligned} & \left[ \left( Z_1^\alpha \tilde{\partial}_\alpha + Y_1^a \dot{\tilde{\partial}}_a \right), \left( Z_2^\beta \tilde{\partial}_\beta + Y_2^b \dot{\tilde{\partial}}_b \right) \right]_{(\rho, \eta) TE} \\ &= \left[ Z_1^\alpha T_a, Z_2^\beta T_\beta \right]_{\pi^*(h^*F)} \oplus \left[ \left( \rho_\alpha^i \circ h \circ \pi \right) Z_1^\alpha \partial_i + Y_1^a \dot{\partial}_a, \right. \\ & \quad \left. \left( \rho_\beta^j \circ h \circ \pi \right) Z_2^\beta \partial_j + Y_2^b \dot{\partial}_b \right]_{TE}, \end{aligned}$$

for any  $\left( Z_1^\alpha \tilde{\partial}_\alpha + Y_1^a \dot{\tilde{\partial}}_a \right)$  and  $\left( Z_2^\beta \tilde{\partial}_\beta + Y_2^b \dot{\tilde{\partial}}_b \right)$ , we obtain that the couple

$$([\cdot]_{(\rho, \eta) TE}, (\tilde{\rho}, Id_E))$$

is a Lie algebroid structure for the vector bundle  $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$ .

The Lie algebroid

$$\left( ((\rho, \eta) TE, (\rho, \eta) \tau_E, E), [\cdot]_{(\rho, \eta) TE}, (\tilde{\rho}, Id_E) \right),$$

is called the *Lie algebroid generalized tangent bundle*.

We consider the  $\mathbf{B}^v$ -morphism  $((\rho, \eta) \pi!, Id_E)$  given by the commutative diagram

$$(3.7) \quad \begin{array}{ccc} (\rho, \eta) TE & \xrightarrow{(\rho, \eta) \pi!} & \pi^*(h^*F) \\ (\rho, \eta) \tau_E \downarrow & & \downarrow pr_1 \\ E & \xrightarrow{id_E} & E \end{array}$$

This is defined as:

$$(3.8) \quad (\rho, \eta) \pi! \left( \left( Z^\alpha \tilde{\partial}_\alpha + Y^a \dot{\tilde{\partial}}_a \right) (u_x) \right) = (Z^\alpha T_\alpha) (u_x),$$

for any  $\left( Z^\alpha \tilde{\partial}_\alpha + Y^a \dot{\tilde{\partial}}_a \right) \in \Gamma((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$ .

Using the  $\mathbf{B}^v$ -morphisms (2.6) and (3.7) we obtain the *tangent*  $(\rho, \eta)$ -application  $((\rho, \eta) T\pi, h \circ \pi)$  of  $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$  source and  $(F, \nu, N)$  target.

**Definition 3.1** The kernel of the tangent  $(\rho, \eta)$ -application is written

$$(V(\rho, \eta) TE, (\rho, \eta) \tau_E, E)$$

and it is called *the vertical subbundle*.

We remark that the set  $\left\{ \dot{\tilde{\partial}}_a, a \in \overline{1, r} \right\}$  is a base of the  $\mathcal{F}(E)$ -module

$$(\Gamma(V(\rho, \eta)TE, (\rho, \eta)\tau_E, E), +, \cdot).$$

**Proposition 3.1** *The short sequence of vector bundles*

$$(3.9) \quad \begin{array}{ccccccccc} 0 & \xrightarrow{i} & V(\rho, \eta)TE & \xrightarrow{i} & (\rho, \eta)TE & \xrightarrow{(\rho, \eta)\pi^!} & \pi^*(h^*F) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ E & \xrightarrow{Id_E} & E & \xrightarrow{Id_E} & E & \xrightarrow{Id_E} & E & \xrightarrow{Id_E} & E \end{array}$$

is exact.

Let  $(\rho, \eta)\Gamma$  be a  $(\rho, \eta)$ -connection for the vector bundle  $(E, \pi, M)$ , i. e. a **Man**-morphism  $(\rho, \eta)\Gamma$  of  $(\rho, \eta)TE$  source and  $V(\rho, \eta)TE$  target defined by

$$(3.10) \quad (\rho, \eta)\Gamma \left( Z^\alpha \tilde{\partial}_\alpha + Y^a \dot{\tilde{\partial}}_a \right) (u_x) = (Y^a + (\rho, \eta)\Gamma_\alpha^a Z^\alpha) \dot{\tilde{\partial}}_a (u_x),$$

so that the **B<sup>V</sup>**-morphism  $((\rho, \eta)\Gamma, Id_E)$  is a split to the left in the previous exact sequence. Its components satisfy the law of transformation

$$(3.11) \quad (\rho, \eta)\Gamma_\gamma^{a'} = M_a^{a'} \circ \pi \left[ \rho_\gamma^i \circ (h \circ \pi) \frac{\partial M_k^a \circ \pi}{\partial x^i} y^b + (\rho, \eta)\Gamma_\gamma^a \right] \Lambda_\gamma^{\gamma'} \circ (h \circ \pi).$$

In the particular case of Lie algebroids,  $(\eta, h) = (Id_M, Id_M)$ , we obtain

$$(3.11)' \quad \rho \Gamma_\gamma^{a'} = M_a^{a'} \circ \pi \left[ \rho_\gamma^i \circ \pi \frac{\partial M_k^a \circ \pi}{\partial x^i} y^b + \rho \Gamma_\gamma^a \right] \Lambda_\gamma^{\gamma'} \circ \pi.$$

In the classical case,  $(\rho, \eta, h) = (Id_{TE}, Id_M, Id_M)$ , we obtain

$$(3.11)'' \quad \Gamma_k^{a'} = M_a^{a'} \circ \pi \left[ \frac{\partial M_k^a \circ \pi}{\partial x^i} y^b + \Gamma_k^a \right] \frac{\partial x^k}{\partial x^{k'}} \circ \pi.$$

The kernel of the **B<sup>V</sup>**-morphism  $((\rho, \eta)\Gamma, Id_E)$  is written  $(H(\rho, \eta)TE, (\rho, \eta)\tau_E, E)$  and is called the *horizontal vector subbundle*.

We put the problem of finding a base for the  $\mathcal{F}(E)$ -module

$$(\Gamma(H(\rho, \eta)TE, (\rho, \eta)\tau_E, E), +, \cdot)$$

of the type

$$\tilde{\delta}_\alpha = Z_\alpha^\beta \tilde{\partial}_\beta + Y_\alpha^a \dot{\tilde{\partial}}_a, \alpha \in \overline{1, r}$$

which satisfies the following conditions:

$$(3.12) \quad \begin{aligned} \Gamma((\rho, \eta)\pi^!, Id_E) \left( \tilde{\delta}_\alpha \right) &= T_\alpha, \\ \Gamma((\rho, \eta)\Gamma, Id_E) \left( \tilde{\delta}_\alpha \right) &= 0. \end{aligned}$$

Then we obtain the sections

$$(3.13) \quad \frac{\delta}{\delta \tilde{z}^\alpha} = \tilde{\partial}_\alpha - (\rho, \eta)\Gamma_\alpha^a \dot{\tilde{\partial}}_a = T_\alpha \oplus \left( (\rho_\alpha^i \circ h \circ \pi) \partial_i - (\rho, \eta)\Gamma_\alpha^a \dot{\partial}_a \right).$$



such that their law of change is a tensorial law under a change of vector fiber charts.

The base  $\left(\tilde{\delta}_\alpha, \dot{\tilde{\partial}}_a\right)$  will be called the *adapted*  $(\rho, \eta)$ -base.

*Remark 3.2* The following equality holds good

$$(3.14) \quad \Gamma(\tilde{\rho}, Id_E) \left(\tilde{\delta}_\alpha\right) = (\rho_\alpha^i \circ h \circ \pi) \partial_i - (\rho, \eta) \Gamma_\alpha^a \dot{\partial}_a.$$

Moreover, if  $(\rho, \eta) \Gamma$  is the  $(\rho, \eta)$ -connection associated to a connection  $\Gamma$  (see [5]), then we obtain

$$(3.15) \quad \Gamma(\tilde{\rho}, Id_E) \left(\tilde{\delta}_\alpha\right) = (\rho_\alpha^i \circ h \circ \pi) \delta_i,$$

where  $\left(\delta_i, \dot{\partial}_a\right)$  is the adapted base for the  $\mathcal{F}(E)$ -module  $(\Gamma(TE, \tau_E, E), +, \cdot)$ .

Let  $(d\tilde{z}^\alpha, d\tilde{y}^b)$  be the natural dual  $(\rho, \eta)$ -base of natural  $(\rho, \eta)$ -base  $\left(\tilde{\delta}_\alpha, \dot{\tilde{\partial}}_a\right)$ .

This is determined by the equations

$$\begin{cases} \langle d\tilde{z}^\alpha, \tilde{\partial}_\beta \rangle = \delta_\beta^\alpha, & \langle d\tilde{z}^\alpha, \dot{\tilde{\partial}}_a \rangle = 0, \\ \langle d\tilde{y}^a, \tilde{\partial}_\beta \rangle = 0, & \langle d\tilde{y}^a, \dot{\tilde{\partial}}_b \rangle = \delta_b^a. \end{cases}$$

We consider the problem of finding a base for the  $\mathcal{F}(E)$ -module

$$(\Gamma((V(\rho, \eta)TE)^*, ((\rho, \eta)\tau_E)^*, E), +, \cdot)$$

of the type

$$\delta\tilde{y}^a = \theta_\alpha^a d\tilde{z}^\alpha + \omega_b^a d\tilde{y}^b, \quad a \in \overline{1, n}$$

which satisfies the following conditions:

$$(3.16) \quad \left\langle \delta\tilde{y}^a, \dot{\tilde{\partial}}_a \right\rangle = 1 \wedge \left\langle \delta\tilde{y}^a, \tilde{\delta}_\alpha \right\rangle = 0.$$

We obtain the sections

$$(3.17) \quad \delta\tilde{y}^a = (\rho, \eta) \Gamma_\alpha^a d\tilde{z}^\alpha + d\tilde{y}^a, \quad a \in \overline{1, n}.$$

such that their changing rule is tensorial under a change of vector fiber charts. The base  $(d\tilde{z}^\alpha, \delta\tilde{y}^a)$  will be called the *adapted dual*  $(\rho, \eta)$ -base.

## 4 Tensor $d$ -fields. Distinguished linear $(\rho, \eta)$ -connections

We consider the following diagram:

$$\begin{array}{ccc} E & & (F, [\cdot, \cdot]_{F,h}, (\rho, \eta)) \\ \pi \downarrow & & \downarrow \nu \\ M & \xrightarrow{h} & N \end{array}$$

where  $(E, \pi, M) \in |\mathbf{B}^V|$  and  $\left( (F, \nu, N), [\cdot, \cdot]_{F,h}, (\rho, \eta) \right)$  is a generalized Lie algebroid.

Let

$$(\mathcal{T}_{q,s}^{p,r}((\rho, \eta) TE, (\rho, \eta) \tau_E, E), +, \cdot)$$

be the  $\mathcal{F}(E)$ -module of tensor fields by  $(\frac{p,r}{q,s})$ -type from the generalized tangent bundle

$$(H(\rho, \eta) TE, (\rho, \eta) \tau_E, E) \oplus (V(\rho, \eta) TE, (\rho, \eta) \tau_E, E).$$

An arbitrarily tensor field  $T$  is written as

$$T = T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \tilde{\delta}_{\alpha_1} \otimes \dots \otimes \tilde{\delta}_{\alpha_p} \otimes d\tilde{z}^{\beta_1} \otimes \dots \otimes d\tilde{z}^{\beta_q} \otimes \tilde{\partial}_{a_1} \otimes \dots \otimes \tilde{\partial}_{a_r} \otimes \delta \tilde{y}^{b_1} \otimes \dots \otimes \delta \tilde{y}^{b_s}.$$

Let

$$(\mathcal{T}((\rho, \eta) TE, (\rho, \eta) \tau_E, E), +, \cdot, \otimes)$$

be the tensor fields algebra of generalized tangent bundle  $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$ .

If  $T_1 \in \mathcal{T}_{q_1, s_1}^{p_1, r_1}((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$  and  $T_2 \in \mathcal{T}_{q_2, s_2}^{p_2, r_2}((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$ , then the components of product tensor field  $T_1 \otimes T_2$  are the products of local components of  $T_1$  and  $T_2$ . Therefore, we obtain  $T_1 \otimes T_2 \in \mathcal{T}_{q_1+q_2, s_1+s_2}^{p_1+p_2, r_1+r_2}((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$ .

Let  $\mathcal{DT}((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$  be the family of tensor fields

$$T \in \mathcal{T}((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$$

for which there exists

$$T_1 \in \mathcal{T}_{q,0}^{p,0}((\rho, \eta) TE, (\rho, \eta) \tau_E, E) \text{ and } T_2 \in \mathcal{T}_{0,s}^{0,r}((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$$

such that  $T = T_1 + T_2$ .

The  $\mathcal{F}(E)$ -module  $(\mathcal{DT}((\rho, \eta) TE, (\rho, \eta) \tau_E, E), +, \cdot)$  will be called the *module of distinguished tensor fields* or the *module of tensor d-fields*.

*Remark 5.1* The elements of

$$\Gamma((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$$

respectively

$$\Gamma(((\rho, \eta) TE)^*, ((\rho, \eta) \tau_E)^*, E)$$

are tensor  $d$ -fields.

**Definition 4.1** Let  $(E, \pi, M)$  be a vector bundle endowed with a  $(\rho, \eta)$ -connection  $(\rho, \eta) \Gamma$  and let

$$(4.1) \quad (X, T) \xrightarrow{(\rho, \eta) D} (\rho, \eta) D_X T$$

be a covariant  $(\rho, \eta)$ -derivative for the tensor algebra of the generalized tangent bundle

$$((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$$

which preserves the horizontal and vertical *IDS* by parallelism. (see [6])

The real local functions

$$((\rho, \eta) H_{\beta\gamma}^\alpha, (\rho, \eta) H_{b\gamma}^a, (\rho, \eta) V_{\beta c}^\alpha, (\rho, \eta) V_{bc}^a)$$

defined by the following equalities:

$$(4.2) \quad \begin{aligned} (\rho, \eta) D_{\tilde{\delta}_\gamma} \tilde{\delta}_\beta &= (\rho, \eta) H_{\beta\gamma}^\alpha \tilde{\delta}_\alpha, & (\rho, \eta) D_{\tilde{\delta}_\gamma} \tilde{\partial}_b &= (\rho, \eta) H_{b\gamma}^a \tilde{\partial}_a \\ (\rho, \eta) D_{\tilde{\partial}_c} \tilde{\delta}_\beta &= (\rho, \eta) V_{\beta c}^\alpha \tilde{\delta}_\alpha, & (\rho, \eta) D_{\tilde{\partial}_c} \tilde{\partial}_b &= (\rho, \eta) V_{bc}^a \tilde{\partial}_a \end{aligned}$$

are the components of a linear  $(\rho, \eta)$ -connection  $((\rho, \eta) H, (\rho, \eta) V)$  for the generalized tangent bundle  $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$  which will be called the *distinguished linear  $(\rho, \eta)$ -connection*.

If  $h = Id_M$ , then the distinguished linear  $(Id_{TM}, Id_M)$ -connection is the classical *distinguished linear connection*.

The components of a distinguished linear connection  $(H, V)$  will be denoted

$$(H_{jk}^i, H_{bk}^a, V_{jc}^i, V_{bc}^a).$$

**Theorem 4.1** *If  $((\rho, \eta)H, (\rho, \eta)V)$  is a distinguished linear  $(\rho, \eta)$ -connection for the generalized tangent bundle  $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$ , then its components satisfy the change relations:*

$$(4.3) \quad \begin{aligned} (\rho, \eta) H_{\beta\gamma}^{\alpha'} &= \Lambda_\alpha^{\alpha'} \circ h \circ \pi \cdot \left[ \Gamma(\tilde{\rho}, Id_E) \left( \tilde{\delta}_\gamma \right) \left( \Lambda_\beta^\alpha \circ h \circ \pi \right) + \right. \\ &\quad \left. + (\rho, \eta) H_{\beta\gamma}^\alpha \cdot \Lambda_\beta^\beta \circ h \circ \pi \right] \cdot \Lambda_\gamma^\gamma \circ h \circ \pi, \\ (\rho, \eta) H_{b\gamma}^{a'} &= M_a^{a'} \circ \pi \cdot \left[ \Gamma(\tilde{\rho}, Id_E) \left( \tilde{\delta}_\gamma \right) (M_b^a \circ \pi) + \right. \\ &\quad \left. + (\rho, \eta) H_{b\gamma}^a \cdot M_b^b \circ \pi \right] \cdot \Lambda_\gamma^\gamma \circ h \circ \pi, \\ (\rho, \eta) V_{\beta c}^{\alpha'} &= \Lambda_\alpha^{\alpha'} \circ h \circ \pi \cdot (\rho, \eta) V_{\beta c}^\alpha \cdot \Lambda_\beta^\beta \circ h \circ \pi \cdot M_c^c \circ \pi, \\ (\rho, \eta) V_{bc}^{a'} &= M_a^{a'} \circ \pi \cdot (\rho, \eta) V_{bc}^a \cdot M_b^b \circ \pi \cdot M_c^c \circ \pi. \end{aligned}$$

**Corollary 4.1** *In the particular case of Lie algebroids,  $(\eta, h) = (Id_M, Id_M)$ , we obtain*

$$(4.3)' \quad \begin{aligned} \rho H_{\beta\gamma}^{\alpha'} &= \Lambda_\alpha^{\alpha'} \circ \pi \cdot \left[ \Gamma(\tilde{\rho}, Id_E) \left( \tilde{\delta}_\gamma \right) \left( \Lambda_\beta^\alpha \circ \pi \right) + \rho H_{\beta\gamma}^\alpha \cdot \Lambda_\beta^\beta \circ \pi \right] \cdot \Lambda_\gamma^\gamma \circ \pi \\ \rho H_{b\gamma}^{a'} &= M_a^{a'} \circ \pi \cdot \left[ \Gamma(\tilde{\rho}, Id_E) \left( \tilde{\delta}_\gamma \right) (M_b^a \circ \pi) + \rho H_{b\gamma}^a \cdot M_b^b \circ \pi \right] \cdot \Lambda_\gamma^\gamma \circ \pi, \\ \rho V_{\beta c}^{\alpha'} &= \Lambda_\alpha^{\alpha'} \circ \pi \cdot \rho V_{\beta c}^\alpha \cdot \Lambda_\beta^\beta \circ \pi \cdot M_c^c \circ \pi, \\ \rho V_{bc}^{a'} &= M_a^{a'} \circ \pi \cdot \rho V_{bc}^a \cdot M_b^b \circ \pi \cdot M_c^c \circ \pi. \end{aligned}$$

*In the classical case,  $(\rho, \eta, h) = (Id_{TE}, Id_M, Id_M)$ , we obtain that the components of a distinguished linear connection  $(H, V)$  verify the change relations:*

$$(4.3)'' \quad \begin{aligned} H_{jk}^{i'} &= \frac{\partial x^{i'}}{\partial x^i} \circ \pi \cdot \left[ \frac{\delta}{\delta x^k} \left( \frac{\partial x^i}{\partial x^j} \circ \pi \right) + H_{jk}^i \cdot \frac{\partial x^j}{\partial x^k} \circ \pi \right] \cdot \frac{\partial x^k}{\partial x^k} \circ \pi, \\ H_{bk}^{a'} &= M_a^{a'} \circ \pi \cdot \left[ \frac{\delta}{\delta x^k} (M_b^a \circ \pi) + H_{bk}^a \cdot M_b^b \circ \pi \right] \cdot \frac{\partial x^k}{\partial x^k} \circ \pi, \\ V_{jc}^{i'} &= \frac{\partial x^{i'}}{\partial x^i} \circ \pi \cdot V_{jc}^i \frac{\partial x^j}{\partial x^i} \circ \pi \cdot M_c^c \circ \pi, \\ V_{bc}^{a'} &= M_a^{a'} \circ \pi \cdot V_{bc}^a \cdot M_b^b \circ \pi \cdot M_c^c \circ \pi. \end{aligned}$$

**Example 4.1** If  $(E, \pi, M)$  is a vector bundle endowed with the  $(\rho, \eta)$ -connection  $(\rho, \eta)\Gamma$ , then the local real functions

$$(4.4) \quad \left( \frac{\partial(\rho, \eta)\Gamma_\gamma^a}{\partial y^b}, \frac{\partial(\rho, \eta)\Gamma_\gamma^a}{\partial y^b}, 0, 0 \right)$$

are the components of a distinguished linear  $(\rho, \eta)$ -connection for  $((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$ , which will be called the *Berwald linear  $(\rho, \eta)$ -connection*.

The Berwald linear  $(Id_{TM}, Id_M)$ -connection are the usual *Berwald linear connection*.

**Theorem 4.2** *If the generalized tangent bundle  $((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$  is endowed with a distinguished linear  $(\rho, \eta)$ -connection  $((\rho, \eta)H, (\rho, \eta)V)$ , then for any*

$$X = Z^\alpha \tilde{\delta}_\alpha + Y^a \tilde{\partial}_a \in \Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$$

and for any

$$T \in \mathcal{T}_{qs}^{pr}((\rho, \eta)TE, (\rho, \eta)\tau_E, E),$$

we obtain the formula:

$$(4.5) \quad \begin{aligned} & (\rho, \eta) D_X \left( T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \tilde{\delta}_{\alpha_1} \otimes \dots \otimes \tilde{\delta}_{\alpha_p} \otimes d\tilde{z}^{\beta_1} \otimes \dots \otimes \right. \\ & \quad \left. \otimes d\tilde{z}^{\beta_q} \otimes \tilde{\partial}_{a_1} \otimes \dots \otimes \tilde{\partial}_{a_r} \otimes \delta \tilde{y}^{b_1} \otimes \dots \otimes \delta \tilde{y}^{b_s} \right) = \\ & = Z^\gamma T_{\beta_1 \dots \beta_q b_1 \dots b_s | \gamma}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \tilde{\delta}_{\alpha_1} \otimes \dots \otimes \tilde{\delta}_{\alpha_p} \otimes d\tilde{z}^{\beta_1} \otimes \dots \otimes d\tilde{z}^{\beta_q} \otimes \tilde{\partial}_{a_1} \otimes \dots \otimes \\ & \quad \otimes \tilde{\partial}_{a_r} \otimes \delta \tilde{y}^{b_1} \otimes \dots \otimes \delta \tilde{y}^{b_s} + Y^c T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} |_c \tilde{\delta}_{\alpha_1} \otimes \dots \otimes \\ & \quad \otimes \tilde{\delta}_{\alpha_p} \otimes d\tilde{z}^{\beta_1} \otimes \dots \otimes d\tilde{z}^{\beta_q} \otimes \tilde{\partial}_{a_1} \otimes \dots \otimes \tilde{\partial}_{a_r} \otimes \delta \tilde{y}^{b_1} \otimes \dots \otimes \delta \tilde{y}^{b_s}, \end{aligned}$$

where

$$(4.6) \quad \begin{aligned} & T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} |_\gamma = \Gamma(\tilde{\rho}, Id_E) \left( \tilde{\delta}_\gamma \right) T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \\ & + (\rho, \eta) H_{\alpha\gamma}^{\alpha_1} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_2 \dots \alpha_p a_1 \dots a_r} + \dots + (\rho, \eta) H_{\alpha\gamma}^{\alpha_p} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_{p-1} a_1 \dots a_r} \\ & - (\rho, \eta) H_{\beta_1 \gamma}^\beta T_{\beta_2 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} - \dots - (\rho, \eta) H_{\beta_q \gamma}^\beta T_{\beta_1 \dots \beta_{q-1} b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \\ & + (\rho, \eta) H_{a\gamma}^{a_1} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_2 \dots a_r} + \dots + (\rho, \eta) H_{a\gamma}^{a_r} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_{r-1} a} \\ & - (\rho, \eta) H_{b_1 \gamma}^b T_{\beta_1 \dots \beta_q b_2 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} - \dots - (\rho, \eta) H_{b_s \gamma}^b T_{\beta_1 \dots \beta_q b_1 \dots b_{s-1} b}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \end{aligned}$$

and

$$(4.7) \quad \begin{aligned} & T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} |_c = \Gamma(\tilde{\rho}, Id_E) \left( \tilde{\partial}_c \right) T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} + \\ & + (\rho, \eta) V_{ac}^{\alpha_1} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_2 \dots \alpha_p a_1 \dots a_r} + \dots + (\rho, \eta) V_{ac}^{\alpha_p} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_{p-1} a_1 \dots a_r} - \\ & - (\rho, \eta) V_{\beta_1 c}^\beta T_{\beta_2 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} - \dots - (\rho, \eta) V_{\beta_q c}^\beta T_{\beta_1 \dots \beta_{q-1} b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} + \\ & + (\rho, \eta) V_{ac}^{a_1} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_2 \dots a_r} + \dots + (\rho, \eta) V_{ac}^{a_r} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_{r-1} a} - \\ & - (\rho, \eta) V_{b_1 c}^b T_{\beta_1 \dots \beta_q b_2 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} - \dots - (\rho, \eta) V_{b_s c}^b T_{\beta_1 \dots \beta_q b_1 \dots b_{s-1} b}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \end{aligned}$$

**Corollary 4.2** *In the particular case of Lie algebroids,  $(\eta, h) = (Id_M, Id_M)$ , we obtain*

$$\begin{aligned}
(4.6)' \quad T_{\beta_1 \dots \beta_q b_1 \dots b_s | \gamma}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} &= \Gamma(\tilde{\rho}, Id_E) \left( \tilde{\delta}_\gamma \right) T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \\
&+ \rho H_{\alpha \gamma}^{\alpha_1} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_2 \dots \alpha_p a_1 \dots a_r} + \dots + \rho H_{\alpha \gamma}^{\alpha_p} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_{p-1} a_1 \dots a_r} \\
&- \rho H_{\beta_1 \gamma}^{\beta} T_{\beta_2 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} - \dots - \rho H_{\beta_q \gamma}^{\beta} T_{\beta_1 \dots \beta_{q-1} b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \\
&+ \rho H_{a \gamma}^{a_1} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_2 \dots a_r} + \dots + \rho H_{a \gamma}^{a_r} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_{r-1} a} \\
&- \rho H_{b_1 \gamma}^b T_{\beta_1 \dots \beta_q b b_2 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} - \dots - \rho H_{b_s \gamma}^b T_{\beta_1 \dots \beta_q b_1 \dots b_{s-1} b}^{\alpha_1 \dots \alpha_p a_1 \dots a_r}
\end{aligned}$$

and

$$\begin{aligned}
(4.7)' \quad T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} |_{c=} &= \Gamma(\tilde{\rho}, Id_E) \left( \tilde{\partial}_c \right) T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} + \\
&+ \rho V_{\alpha c}^{\alpha_1} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_2 \dots \alpha_p a_1 \dots a_r} + \dots + \rho V_{\alpha c}^{\alpha_p} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_{p-1} a_1 \dots a_r} - \\
&- \rho V_{\beta_1 c}^{\beta} T_{\beta_2 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} - \dots - \rho V_{\beta_q c}^{\beta} T_{\beta_1 \dots \beta_{q-1} b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} + \\
&+ \rho V_{a c}^{a_1} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_2 \dots a_r} + \dots + \rho V_{a c}^{a_r} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_{r-1} a} - \\
&- \rho V_{b_1 c}^b T_{\beta_1 \dots \beta_q b b_2 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} - \dots - \rho V_{b_s c}^b T_{\beta_1 \dots \beta_q b_1 \dots b_{s-1} b}^{\alpha_1 \dots \alpha_p a_1 \dots a_r}.
\end{aligned}$$

In the classical case,  $(\rho, \eta, h) = (Id_{TE}, Id_M, Id_M)$ , we obtain

$$\begin{aligned}
(4.6)'' \quad T_{j_1 \dots j_q b_1 \dots b_s | k}^{i_1 \dots i_p a_1 \dots a_r} &= \delta_k \left( T_{j_1 \dots j_q b_1 \dots b_s}^{i_1 \dots i_p a_1 \dots a_r} \right) \\
&+ H_{ik}^{i_1} T_{j_1 \dots j_q b_1 \dots b_s}^{i_2 \dots i_p a_1 \dots a_r} + \dots + H_{ik}^{i_p} T_{j_1 \dots j_q b_1 \dots b_s}^{i_1 \dots i_{p-1} a_1 \dots a_r} \\
&- H_{j_1 k}^j T_{j_2 \dots j_q b_1 \dots b_s}^{i_1 \dots i_p a_1 \dots a_r} - \dots - H_{j_q k}^j T_{j_1 \dots j_{q-1} b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \\
&+ H_{ak}^{a_1} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_2 \dots a_r} + \dots + H_{ak}^{a_r} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_{r-1} a} \\
&- H_{b_1 k}^b T_{\beta_1 \dots \beta_q b b_2 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} - \dots - H_{b_s k}^b T_{\beta_1 \dots \beta_q b_1 \dots b_{s-1} b}^{\alpha_1 \dots \alpha_p a_1 \dots a_r}
\end{aligned}$$

and

$$\begin{aligned}
(4.7)'' \quad T_{j_1 \dots j_q b_1 \dots b_s}^{i_1 \dots i_p a_1 \dots a_r} |_{c=} &= \dot{\partial}_c \left( T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \right) + \\
&+ V_{ic}^{i_1} T_{j_1 \dots j_q b_1 \dots b_s}^{i_2 \dots i_p a_1 \dots a_r} + \dots + V_{ic}^{i_p} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{i_1 \dots i_{p-1} a_1 \dots a_r} - \\
&- V_{j_1 c}^j T_{j_2 \dots j_q b_1 \dots b_s}^{i_1 \dots i_p a_1 \dots a_r} - \dots - V_{j_q c}^j T_{j_1 \dots j_{q-1} b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} + \\
&+ V_{ac}^{a_1} T_{j_1 \dots j_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_2 \dots a_r} + \dots + V_{ac}^{a_r} T_{j_1 \dots j_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_{r-1} a} - \\
&- V_{b_1 c}^b T_{j_1 \dots j_q b b_2 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} - \dots - V_{b_s c}^b T_{j_1 \dots j_q b_1 \dots b_{s-1} b}^{\alpha_1 \dots \alpha_p a_1 \dots a_r}.
\end{aligned}$$

**Definition 4.2** If  $(E, \pi, M) = (F, \nu, N)$ ,  $(\rho, \eta) \Gamma$  is a  $(\rho, \eta)$ -connection for the vector bundle  $(E, \pi, M)$  and

$$\left( (\rho, \eta) H_{bc}^a, (\rho, \eta) \tilde{H}_{bc}^a, (\rho, \eta) V_{bc}^a, (\rho, \eta) \tilde{V}_{bc}^a \right)$$

are the components of a distinguished linear  $(\rho, \eta)$ -connection for the generalized tangent bundle  $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$  such that

$$(\rho, \eta) H_{bc}^a = (\rho, \eta) \tilde{H}_{bc}^a \text{ and } (\rho, \eta) V_{bc}^a = (\rho, \eta) \tilde{V}_{bc}^a,$$

then we will say that *the generalized tangent bundle  $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$  is endowed with a normal distinguished linear  $(\rho, \eta)$ -connection on components  $((\rho, \eta) H_{bc}^a, (\rho, \eta) V_{bc}^a)$ .*

In the particular case of Lie algebroids,  $(\eta, h) = (Id_M, Id_M)$ , the components of a normal distinguished linear  $(\rho, Id_M)$ -connection  $(\rho H, \rho V)$  will be denoted  $(\rho H_{bc}^a, \rho V_{bc}^a)$ .

In the classical case,  $(\rho, \eta, h) = (Id_{TE}, Id_M, Id_M)$ , the components of a normal distinguished linear  $(Id_{TM}, Id_M)$ -connection  $(H, V)$  will be denoted  $(H_{jk}^i, V_{jk}^i)$ .

## 5 The $(\rho, \eta)$ -(pseudo)metrizability

We consider the following diagram:

$$\begin{array}{ccc} E & & (F, [, ]_{F,h}, (\rho, \eta)) \\ \pi \downarrow & & \downarrow \nu \\ M & \xrightarrow{h} & N \end{array}$$

where  $(E, \pi, M) \in |\mathbf{B}^V|$  and  $((F, \nu, N), [, ]_{F,h}, (\rho, \eta))$  is a generalized Lie algebroid. Let  $(\rho, \eta) \Gamma$  be a  $(\rho, \eta)$ -connection for the vector bundle  $(E, \pi, M)$  and let  $((\rho, \eta) H, (\rho, \eta) V)$  be a distinguished linear  $(\rho, \eta)$ -connection for the generalized tangent bundle  $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$ .

**Definition 5.1** A tensor  $d$ -field

$$G = g_{\alpha\beta} d\tilde{z}^\alpha \otimes d\tilde{z}^\beta + g_{ab} \delta\tilde{y}^a \otimes \delta\tilde{y}^b \in \mathcal{DT}_{22}^{00}((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$$

will be called *pseudometrical structure* if its components are symmetric and the matrices  $\|g_{\alpha\beta}(u_x)\|$  and  $\|g_{ab}(u_x)\|$  are nondegenerate, for any point  $u_x \in E$ .

Moreover, if the matrices  $\|g_{\alpha\beta}(u_x)\|$  and  $\|g_{ab}(u_x)\|$  has constant signature, then the tensor  $d$ -field  $G$  will be called *metrical structure*.

Let

$$G = g_{\alpha\beta} d\tilde{z}^\alpha \otimes d\tilde{z}^\beta + g_{ab} \delta\tilde{y}^a \otimes \delta\tilde{y}^b$$

be a (pseudo)metrical structure. If  $\alpha, \beta \in \overline{1, p}$  and  $a, b \in \overline{1, r}$ , then for any vector local  $(m+r)$ -chart  $(U, s_U)$  of  $(E, \pi, M)$ , we consider the real functions

$$\pi^{-1}(U) \xrightarrow{\tilde{g}^{\beta\alpha}} \mathbb{R}$$

and

$$\pi^{-1}(U) \xrightarrow{\tilde{g}^{ba}} \mathbb{R}$$

such that

$$\|\tilde{g}^{\beta\alpha}(u_x)\| = \|g_{\alpha\beta}(u_x)\|^{-1}$$

and

$$\|\tilde{g}^{ba}(u_x)\| = \|g_{ab}(u_x)\|^{-1},$$

for any  $u_x \in \pi^{-1}(U) \setminus \{0_x\}$ .

**Definition 5.2** If around each point  $x \in M$  it exists a local vector  $m + r$ -chart  $(U, s_U)$  and a local  $m$ -chart  $(U, \xi_U)$  such that  $g_{\alpha\beta} \circ s_U^{-1} \circ (\xi_U^{-1} \times Id_{\mathbb{R}^m})(x, y)$  and  $g_{ab} \circ s_U^{-1} \circ (\xi_U^{-1} \times Id_{\mathbb{R}^m})(x, y)$  depends only on  $x$ , for any  $u_x \in \pi^{-1}(U)$ , then we will say that the (pseudo)metrical structure

$$G = g_{\alpha\beta} d\tilde{z}^\alpha \otimes d\tilde{z}^\beta + g_{ab} \delta\tilde{y}^a \otimes \delta\tilde{y}^b$$

is a Riemannian (pseudo)metrical structure.

If only the condition is verified:

" $g_{\alpha\beta} \circ s_U^{-1} \circ (\xi_U^{-1} \times Id_{\mathbb{R}^m})(x, y)$  depends only on  $x$ , for any  $u_x \in \pi^{-1}(U)$ " respectively " $g_{ab} \circ s_U^{-1} \circ (\xi_U^{-1} \times Id_{\mathbb{R}^m})(x, y)$  depends only on  $x$ , for any  $u_x \in \pi^{-1}(U)$ ", then we will say that the (pseudo)metrical structure  $G$  is a Riemannian  $\mathcal{H}$ -(pseudo)metrical structure respectively a Riemannian  $\mathcal{V}$ -(pseudo)metrical structure.

**Definition 5.3** If around each point  $x \in M$  there exists a local vector  $m + r$ -chart  $(U, s_U)$  and a local  $m$ -chart  $(U, \xi_U)$  such that  $g_{\alpha\beta} \circ s_U^{-1} \circ (\xi_U^{-1} \times Id_{\mathbb{R}^m})(x, y)$  and  $g_{ab} \circ s_U^{-1} \circ (\xi_U^{-1} \times Id_{\mathbb{R}^m})(x, y)$  depends only on  $y$ , for any  $u_x \in \pi^{-1}(U)$ , then we will say that the (pseudo)metrical structure

$$G = g_{\alpha\beta} d\tilde{z}^\alpha \otimes d\tilde{z}^\beta + g_{ab} \delta\tilde{y}^a \otimes \delta\tilde{y}^b$$

is a locally Minkowski structure.

If only the condition is verified:

" $g_{\alpha\beta} \circ s_U^{-1} \circ (\xi_U^{-1} \times Id_{\mathbb{R}^m})(x, y)$  depends only on  $y$ , for any  $u_x \in \pi^{-1}(U)$ " respectively " $g_{ab} \circ s_U^{-1} \circ (\xi_U^{-1} \times Id_{\mathbb{R}^m})(x, y)$  depends only on  $y$ , for any  $u_x \in \pi^{-1}(U)$ ", then we will say that the (pseudo)metrical structure  $G$  is a (pseudo)metrical structure  $\mathcal{H}$ -locally Minkowski or  $\mathcal{V}$ -locally Minkowski, respectively.

**Definition 5.4** If there exists a (pseudo)metrical structure

$$G = g_{\alpha\beta} d\tilde{z}^\alpha \otimes d\tilde{z}^\beta + g_{ab} \delta\tilde{y}^a \otimes \delta\tilde{y}^b$$

and a distinguished linear  $(\rho, \eta)$ -connection

$$((\rho, \eta) H, (\rho, \eta) V)$$

such that

$$(5.1) \quad (\rho, \eta) D_X G = 0, \quad \forall X \in \Gamma((\rho, \eta) TE, (\rho, \eta) \tau_E, E).$$

then the generalized tangent bundle  $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$  will be called  $(\rho, \eta)$ -(pseudo)metrizable

Condition (5.1) is equivalent with the following equalities:

$$(5.2) \quad g_{\alpha\beta}|_\gamma = 0, \quad g_{ab}|_\gamma = 0, \quad g_{\alpha\beta}|_c = 0, \quad g_{ab}|_c = 0.$$

If  $g_{\alpha\beta}|_\gamma = 0$  and  $g_{ab}|_\gamma = 0$ , then we will say that the vector bundle  $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$  is  $\mathcal{H}$ -( $\rho, \eta$ )-(pseudo)metrizable.

If  $g_{\alpha\beta}|_c = 0$  and  $g_{ab}|_c = 0$ , then we will say that the vector bundle  $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$  is  $\mathcal{V}$ -( $\rho, \eta$ )-(pseudo)metrizable.

**Theorem 5.1** If  $\left( (\rho, \eta) \overset{0}{H}, (\rho, \eta) \overset{0}{V} \right)$  is a distinguished linear  $(\rho, \eta)$ -connection for the generalized tangent bundle  $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$  and  $G = g_{\alpha\beta} d\tilde{z}^\alpha \otimes d\tilde{z}^\beta + g_{ab} \delta\tilde{y}^a \otimes \delta\tilde{y}^b$

is a (pseudo)metrical structure, then the following real local functions:

$$\begin{aligned}
(\rho, \eta) H_{\beta\gamma}^\alpha &= \frac{1}{2} \tilde{g}^{\alpha\varepsilon} \left( \Gamma(\tilde{\rho}, Id_E) \left( \tilde{\delta}_\gamma \right) g_{\varepsilon\beta} + \Gamma(\tilde{\rho}, Id_E) \left( \tilde{\delta}_\beta \right) g_{\varepsilon\gamma} - \Gamma(\tilde{\rho}, Id_E) \left( \tilde{\delta}_\varepsilon \right) g_{\beta\gamma} \right. \\
&\quad \left. + g_{\theta\varepsilon} L_{\gamma\beta}^\theta \circ h \circ \pi - g_{\beta\theta} L_{\gamma\varepsilon}^\theta \circ h \circ \pi - g_{\theta\gamma} L_{\beta\varepsilon}^\theta \circ h \circ \pi \right), \\
(5.3) \quad (\rho, \eta) H_{b\gamma}^a &= (\rho, \eta) \overset{0}{H}_{b\gamma}^a + \frac{1}{2} \tilde{g}^{ac} g_{bc| \gamma}^0, \\
(\rho, \eta) V_{\beta c}^\alpha &= (\rho, \eta) \overset{0}{V}_{\beta c}^\alpha + \frac{1}{2} \tilde{g}^{\alpha\varepsilon} g_{\beta\varepsilon| c}^0, \\
(\rho, \eta) V_{bc}^a &= \frac{1}{2} \tilde{g}^{ae} \left( \Gamma(\tilde{\rho}, Id_E) \left( \dot{\tilde{\partial}}_c \right) g_{eb} + \Gamma(\tilde{\rho}, Id_E) \left( \dot{\tilde{\partial}}_b \right) g_{ec} - \Gamma(\tilde{\rho}, Id_E) \left( \dot{\tilde{\partial}}_e \right) g_{bc} \right)
\end{aligned}$$

are components of a distinguished linear  $(\rho, \eta)$ -connection such that the generalized tangent bundle  $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$  becomes  $(\rho, \eta)$ -(pseudo)metrizable.

**Corollary 5.1** In the particular case of Lie algebroids,  $(\eta, h) = (Id_M, Id_M)$ , then we obtain

$$\begin{aligned}
\rho H_{\beta\gamma}^\alpha &= \frac{1}{2} \tilde{g}^{\alpha\varepsilon} \left( \Gamma(\tilde{\rho}, Id_E) \left( \tilde{\delta}_\gamma \right) g_{\varepsilon\beta} + \Gamma(\tilde{\rho}, Id_E) \left( \tilde{\delta}_\beta \right) g_{\varepsilon\gamma} - \Gamma(\tilde{\rho}, Id_E) \left( \tilde{\delta}_\varepsilon \right) g_{\beta\gamma} \right. \\
&\quad \left. + g_{\theta\varepsilon} L_{\gamma\beta}^\theta \circ \pi - g_{\beta\theta} L_{\gamma\varepsilon}^\theta \circ \pi - g_{\theta\gamma} L_{\beta\varepsilon}^\theta \circ \pi \right), \\
(5.3)' \quad \rho H_{b\gamma}^a &= \rho \overset{0}{H}_{b\gamma}^a + \frac{1}{2} \tilde{g}^{ac} g_{bc| \gamma}^0, \\
\rho V_{\beta c}^\alpha &= \rho \overset{0}{V}_{\beta c}^\alpha + \frac{1}{2} \tilde{g}^{\alpha\varepsilon} g_{\beta\varepsilon| c}^0, \\
\rho V_{bc}^a &= \frac{1}{2} \tilde{g}^{ae} \left( \Gamma(\tilde{\rho}, Id_E) \left( \dot{\tilde{\partial}}_c \right) g_{eb} + \Gamma(\tilde{\rho}, Id_E) \left( \dot{\tilde{\partial}}_b \right) g_{ec} - \Gamma(\tilde{\rho}, Id_E) \left( \dot{\tilde{\partial}}_e \right) g_{bc} \right)
\end{aligned}$$

In the classicale case,  $(\rho, \eta, h) = (Id_{TE}, Id_M, Id_M)$ , then we obtain

$$\begin{aligned}
(5.3)'' \quad H_{jk}^i &= \frac{1}{2} \tilde{g}^{ih} (\delta_k g_{hj} + \delta_j g_{hk} - \delta_h g_{jk}) \\
H_{bk}^a &= \overset{0}{H}_{bk}^a + \frac{1}{2} \tilde{g}^{ac} g_{bc| k}^0, \\
V_{jc}^i &= \overset{0}{V}_{jc}^i + \frac{1}{2} \tilde{g}^{ih} g_{jh| c}^0, \\
V_{bc}^a &= \frac{1}{2} \tilde{g}^{ae} \left( \dot{\partial}_c g_{eb} + \dot{\partial}_b g_{ec} - \dot{\partial}_e g_{bc} \right)
\end{aligned}$$

**Theorem 5.2** If the distinguished linear  $(\rho, \eta)$ -connection  $\left( (\rho, \eta) \overset{0}{H}, (\rho, \eta) \overset{0}{V} \right)$  coincides with the Berwald linear  $(\rho, \eta)$ -connection in the previous theorem, then the local real



functions:

$$\begin{aligned}
(\rho, \eta) \overset{c}{H}_{\beta\gamma}^{\alpha} &= \frac{1}{2} \tilde{g}^{\alpha\epsilon} \left( \Gamma(\tilde{\rho}, Id_E) \left( \tilde{\delta}_\gamma \right) g_{\epsilon\beta} + \Gamma(\tilde{\rho}, Id_E) \left( \tilde{\delta}_\beta \right) g_{\epsilon\gamma} \right. \\
&\quad \left. - \Gamma(\tilde{\rho}, Id_E) \left( \tilde{\delta}_\epsilon \right) g_{\beta\gamma} + g_{\theta\epsilon} L_{\gamma\beta}^\theta \circ h \circ \pi, \right. \\
&\quad \left. - g_{\beta\theta} L_{\gamma\epsilon}^\theta \circ h \circ \pi - g_{\theta\gamma} L_{\beta\epsilon}^\theta \circ h \circ \pi \right), \\
(\rho, \eta) \overset{c}{H}_{b\gamma}^a &= \frac{\partial(\rho, \eta) \Gamma_\gamma^a}{\partial y^b} + \frac{1}{2} \tilde{g}^{ac} g_{bc|_\gamma}^0, \\
(\rho, \eta) \overset{c}{V}_{\beta c}^{\alpha} &= \frac{1}{2} \tilde{g}^{\alpha\epsilon} \frac{\partial g_{\beta\epsilon}}{\partial y^c}, \\
(\rho, \eta) \overset{c}{V}_{bc}^a &= \frac{1}{2} \tilde{g}^{ae} \left( \frac{\partial g_{e\beta}}{\partial y^c} + \frac{\partial g_{ec}}{\partial y^b} - \frac{\partial g_{bc}}{\partial y^e} \right)
\end{aligned} \tag{5.4}$$

are the components of a distinguished linear  $(\rho, \eta)$ -connection such that the generalized tangent bundle  $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$  becomes  $(\rho, \eta)$ -(pseudo)metrizable.

Moreover, if the (pseudo)metrical structure  $G$  is  $\mathcal{H}$ - and  $\mathcal{V}$ -Riemannian, then the local real functions:

$$\begin{aligned}
(\rho, \eta) \overset{c}{H}_{\beta\gamma}^{\alpha} &= \frac{1}{2} \tilde{g}^{\alpha\epsilon} \left( \rho_\gamma^k \circ h \circ \pi \frac{\partial g_{\epsilon\beta}}{\partial x^k} + \rho_\beta^j \circ h \circ \pi \frac{\partial g_{\epsilon\gamma}}{\partial x^j} - \rho_\epsilon^e \circ h \circ \pi \frac{\partial g_{\beta\gamma}}{\partial x^e} \right. \\
&\quad \left. + g_{\theta\epsilon} L_{\gamma\beta}^\theta \circ h \circ \pi - g_{\beta\theta} L_{\gamma\epsilon}^\theta \circ h \circ \pi - g_{\theta\gamma} L_{\beta\epsilon}^\theta \circ h \circ \pi \right), \\
(\rho, \eta) \overset{c}{H}_{b\gamma}^a &= \frac{\partial(\rho, \eta) \Gamma_\gamma^a}{\partial y^b} + \frac{1}{2} \tilde{g}^{ac} \left( \rho_\gamma^i \circ h \circ \pi \frac{\partial g_{bc}}{\partial x^i} - \frac{\partial(\rho, \eta) \Gamma_\gamma^e}{\partial y^b} g_{ec} - \frac{\partial(\rho, \eta) \Gamma_\gamma^e}{\partial y^c} g_{eb} \right), \\
(\rho, \eta) \overset{c}{V}_{\beta c}^{\alpha} &= 0, \\
(\rho, \eta) \overset{c}{V}_{bc}^a &= 0.
\end{aligned} \tag{5.5}$$

are the components of a distinguished linear  $(\rho, \eta)$ -connection such that the generalized tangent bundle  $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$  becomes  $(\rho, \eta)$ -(pseudo)metrizable.

**Corollary 5.2** In the particular case of Lie algebroids,  $(\eta, h) = (Id_M, Id_M)$ , then we obtain

$$\begin{aligned}
\rho \overset{c}{H}_{\beta\gamma}^{\alpha} &= \frac{1}{2} \tilde{g}^{\alpha\epsilon} \left( \Gamma(\tilde{\rho}, Id_E) \left( \tilde{\delta}_\gamma \right) g_{\epsilon\beta} + \Gamma(\tilde{\rho}, Id_E) \left( \tilde{\delta}_\beta \right) g_{\epsilon\gamma} \right. \\
&\quad \left. - \Gamma(\tilde{\rho}, Id_E) \left( \tilde{\delta}_\epsilon \right) g_{\beta\gamma} + g_{\theta\epsilon} L_{\gamma\beta}^\theta \circ \pi - g_{\beta\theta} L_{\gamma\epsilon}^\theta \circ \pi - g_{\theta\gamma} L_{\beta\epsilon}^\theta \circ \pi \right) \\
\rho \overset{c}{H}_{b\gamma}^a &= \frac{\partial \rho \Gamma_\gamma^a}{\partial y^b} + \frac{1}{2} \tilde{g}^{ac} g_{bc|_\gamma}^0, \\
\rho \overset{c}{V}_{\beta c}^{\alpha} &= \frac{1}{2} \tilde{g}^{\alpha\epsilon} \frac{\partial g_{\beta\epsilon}}{\partial y^c}, \\
\rho \overset{c}{V}_{bc}^a &= \frac{1}{2} \tilde{g}^{ae} \left( \frac{\partial g_{e\beta}}{\partial y^c} + \frac{\partial g_{ec}}{\partial y^b} - \frac{\partial g_{bc}}{\partial y^e} \right)
\end{aligned} \tag{5.4}'$$

If the (pseudo)metrical structure  $G$  is  $\mathcal{H}$ - and  $\mathcal{V}$ -Riemannian, then

$$\begin{aligned}
(5.5)' \quad \rho \overset{c}{H}_{\beta\gamma}^{\alpha} &= \frac{1}{2} \tilde{g}^{\alpha\varepsilon} \left( \rho_{\gamma}^k \circ \pi \frac{\partial g_{\varepsilon\beta}}{\partial x^k} + \rho_{\beta}^j \circ \pi \frac{\partial g_{\varepsilon\gamma}}{\partial x^j} - \rho_{\varepsilon}^e \circ \pi \frac{\partial g_{\beta\gamma}}{\partial x^e} + \right. \\
&\quad \left. + g_{\theta\varepsilon} L_{\gamma\beta}^{\theta} \circ \pi - g_{\beta\theta} L_{\gamma\varepsilon}^{\theta} \circ \pi - g_{\theta\gamma} L_{\beta\varepsilon}^{\theta} \circ \pi \right), \\
\rho \overset{c}{H}_{b\gamma}^a &= \frac{\partial \rho_{\gamma}^a}{\partial y^b} + \frac{1}{2} \tilde{g}^{ac} \left( \rho_{\gamma}^i \circ \pi \frac{\partial g_{bc}}{\partial x^i} - \frac{\partial \rho_{\gamma}^e}{\partial y^b} g_{ec} - \frac{\partial \rho_{\gamma}^e}{\partial y^c} g_{eb} \right), \\
\rho \overset{c}{V}_{\beta c}^{\alpha} &= 0, \quad \rho \overset{c}{V}_{bc}^a = 0
\end{aligned}$$

In the classicale case,  $(\rho, \eta, h) = (Id_{TE}, Id_M, Id_M)$ , then we obtain

$$\begin{aligned}
(5.4)'' \quad \overset{c}{H}_{jk}^i &= \frac{1}{2} \tilde{g}^{ih} (\delta_k g_{hj} + \delta_j g_{hk} - \delta_h g_{jk}) \\
\overset{c}{H}_{bk}^a &= \frac{\partial \Gamma_k^a}{\partial y^b} + \frac{1}{2} \tilde{g}^{ac} g_{bc|k}^0, \\
\overset{c}{V}_{jc}^i &= \frac{1}{2} \tilde{g}^{ih} \frac{\partial g_{jh}}{\partial y^c}, \\
\overset{c}{V}_{bc}^a &= \frac{1}{2} \tilde{g}^{ae} \left( \frac{\partial g_{e\beta}}{\partial y^c} + \frac{\partial g_{ec}}{\partial y^b} - \frac{\partial g_{bc}}{\partial y^e} \right)
\end{aligned}$$

If the (pseudo)metrical structure  $G$  is  $\mathcal{H}$ - and  $\mathcal{V}$ -Riemannian, then

$$\begin{aligned}
(5.5)'' \quad \overset{c}{H}_{jk}^i &= \frac{1}{2} \tilde{g}^{ih} \left( \frac{\partial g_{hj}}{\partial x^k} + \frac{\partial g_{hk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^h} \right) \\
\overset{c}{H}_{bk}^a &= \frac{\partial \Gamma_k^a}{\partial y^b} + \frac{1}{2} \tilde{g}^{ac} \left( \frac{\partial g_{bc}}{\partial x^i} - \frac{\partial \Gamma_k^e}{\partial y^b} g_{ec} - \frac{\partial \Gamma_k^e}{\partial y^c} g_{eb} \right), \\
\overset{c}{V}_{jc}^i &= 0, \quad \overset{c}{V}_{bc}^a = 0
\end{aligned}$$

**Theorem 5.3** Let  $(\rho, \eta) \Gamma$  be a  $(\rho, \eta)$ -connection for the vector bundle  $(E, \pi, M)$ . Let

$$\left( (\rho, \eta) \overset{0}{H}, (\rho, \eta) \overset{0}{V} \right)$$

be a distinguished linear  $(\rho, \eta)$ -connection for  $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$  and let

$$G = g_{\alpha\beta} d\tilde{z}^{\alpha} \otimes d\tilde{z}^{\beta} + g_{ab} \delta\tilde{y}^a \otimes \delta\tilde{y}^b$$

be a (pseudo)metrical structure.

Let

$$\begin{aligned}
(5.6) \quad O_{\beta\gamma}^{\alpha\varepsilon} &= \frac{1}{2} (\delta_{\beta}^{\alpha} \delta_{\gamma}^{\varepsilon} - g_{\beta\gamma} \tilde{g}^{\alpha\varepsilon}), \quad O_{\beta\gamma}^{*\alpha\varepsilon} = \frac{1}{2} (\delta_{\beta}^{\alpha} \delta_{\gamma}^{\varepsilon} + g_{\beta\gamma} \tilde{g}^{\alpha\varepsilon}), \\
O_{bc}^{ae} &= \frac{1}{2} (\delta_b^a \delta_c^e - g_{bc} \tilde{g}^{ae}), \quad O_{bc}^{*ae} = \frac{1}{2} (\delta_b^a \delta_c^e + g_{bc} \tilde{g}^{ae}),
\end{aligned}$$

be the Obata operators.

If the real local functions  $X_{\beta\gamma}^\alpha, X_{\beta c}^\alpha, Y_{b\gamma}^a, Y_{bc}^a$  are components of tensor fields, then the local real functions given in the following:

$$\begin{aligned}
(\rho, \eta) H_{\beta\gamma}^\alpha &= (\rho, \eta) \overset{c}{H}_{\beta\gamma}^\alpha + O_{\gamma\eta}^{\alpha\varepsilon} X_{\varepsilon\beta}^\eta, \\
(\rho, \eta) H_{b\gamma}^a &= (\rho, \eta) \overset{c}{H}_{b\gamma}^a + O_{bd}^{ae} Y_{e\gamma}^d, \\
(\rho, \eta) V_{\beta c}^\alpha &= (\rho, \eta) \overset{c}{V}_{\beta c}^\alpha + O_{\beta\eta}^{*\alpha\varepsilon} X_{\varepsilon c}^\eta, \\
(\rho, \eta) V_{bc}^a &= (\rho, \eta) \overset{c}{V}_{bc}^a + O_{bd}^{*ae} Y_{ec}^d,
\end{aligned}
\tag{5.7}$$

are the components of a distinguished linear  $(\rho, \eta)$ -connection such that the generalized tangent bundle  $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$  becomes  $(\rho, \eta)$ -(pseudo)metrizable.

**Corollary 5.2** In the particular case of Lie algebroids,  $(\eta, h) = (Id_M, Id_M)$ , then we obtain

$$\begin{aligned}
\rho H_{\beta\gamma}^\alpha &= \rho \overset{c}{H}_{\beta\gamma}^\alpha + O_{\gamma\eta}^{\alpha\varepsilon} X_{\varepsilon\beta}^\eta, \\
\rho H_{b\gamma}^a &= \rho \overset{c}{H}_{b\gamma}^a + O_{bd}^{ae} Y_{e\gamma}^d, \\
\rho V_{\beta c}^\alpha &= \rho \overset{c}{V}_{\beta c}^\alpha + O_{\beta\eta}^{*\alpha\varepsilon} X_{\varepsilon c}^\eta, \\
\rho V_{bc}^a &= \rho \overset{c}{V}_{bc}^a + O_{bd}^{*ae} Y_{ec}^d,
\end{aligned}
\tag{5.7}'$$

In the classicale case,  $(\rho, \eta, h) = (Id_{TE}, Id_M, Id_M)$ , then we obtain (see [16])

$$\begin{aligned}
H_{jk}^i &= \overset{c}{H}_{jk}^i + O_{kh}^{il} X_{lj}^h, \\
H_{bk}^a &= \overset{c}{H}_{bk}^a + O_{bd}^{ae} Y_{ek}^d, \\
V_{jc}^i &= \overset{c}{V}_{jc}^i + O_{jh}^{*il} X_{lc}^h, \\
\rho V_{bc}^a &= \overset{c}{V}_{bc}^a + O_{bd}^{*ae} Y_{ec}^d,
\end{aligned}
\tag{5.7}''$$

**Theorem 5.3** Let  $(\rho, \eta) \Gamma$  be a  $(\rho, \eta)$ -connection for the vector bundle  $(E, \pi, M)$ .

If

$$\left( (\rho, \eta) \overset{0}{H}, (\rho, \eta) \overset{0}{V} \right)$$

is a distinguished linear  $(\rho, \eta)$ -connection for the generalized tangent bundle

$$((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$$

and

$$G = g_{\alpha\beta} d\tilde{z}^\alpha \otimes d\tilde{z}^\beta + g_{ab} \delta\tilde{y}^a \otimes \delta\tilde{y}^b$$

is a (pseudo)metrical structure, then the real local functions:

$$\begin{aligned}
(\rho, \eta) H_{\beta\gamma}^\alpha &= (\rho, \eta) \overset{0}{H}_{\beta\gamma}^\alpha + \frac{1}{2} \tilde{g}^{\alpha\varepsilon} g_{\varepsilon\beta|_\gamma}^0, \\
(\rho, \eta) H_{b\gamma}^a &= (\rho, \eta) \overset{0}{H}_{b\gamma}^a + \frac{1}{2} \tilde{g}^{ae} g_{eb|_\gamma}^0, \\
(\rho, \eta) V_{\beta c}^\alpha &= (\rho, \eta) \overset{0}{V}_{\beta c}^\alpha + \frac{1}{2} \tilde{g}^{\alpha\varepsilon} g_{\varepsilon\beta}^0|_c, \\
(\rho, \eta) V_{bc}^a &= (\rho, \eta) \overset{0}{V}_{bc}^a + \frac{1}{2} \tilde{g}^{ae} g_{eb}^0|_c
\end{aligned}
\tag{5.8}$$

are the components of a distinguished linear  $(\rho, \eta)$ -connection such that the generalized tangent bundle  $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$  becomes  $(\rho, \eta)$ -(pseudo)metrizable.

**Corollary 5.3** *In the particular case of Lie algebroids,  $(\eta, h) = (Id_M, Id_M)$ , then we obtain*

$$\begin{aligned}
 \rho H_{\beta\gamma}^\alpha &= \rho H_{\beta\gamma}^\alpha + \frac{1}{2} \tilde{g}^{\alpha\varepsilon} g_{\varepsilon\beta|_\gamma}^0, \\
 \rho H_{b\gamma}^a &= \rho H_{b\gamma}^a + \frac{1}{2} \tilde{g}^{ae} g_{eb|_\gamma}^0, \\
 \rho V_{\beta c}^\alpha &= \rho V_{\beta c}^\alpha + \frac{1}{2} \tilde{g}^{\alpha\varepsilon} g_{\varepsilon\beta}^0|_c, \\
 \rho V_{bc}^a &= \rho V_{bc}^a + \frac{1}{2} \tilde{g}^{ae} g_{eb}^0|_c
 \end{aligned}
 \tag{5.8}'$$

In the classical case,  $(\rho, \eta, h) = (Id_{TE}, Id_M, Id_M)$ , then we obtain (see [14])

$$\begin{aligned}
 H_{jk}^i &= H_{jk}^i + \frac{1}{2} \tilde{g}^{ih} g_{hj|k}^0, \\
 H_{bk}^a &= H_{bk}^a + \frac{1}{2} \tilde{g}^{ae} g_{eb|k}^0, \\
 V_{jc}^i &= V_{jc}^i + \frac{1}{2} \tilde{g}^{ih} g_{hj}^0|_c, \\
 V_{bc}^a &= V_{bc}^a + \frac{1}{2} \tilde{g}^{ae} g_{eb}^0|_c
 \end{aligned}
 \tag{5.8}''$$

## 6 Generalized Lagrange $(\rho, \eta)$ -spaces, Lagrange $(\rho, \eta)$ -spaces and Finsler $(\rho, \eta)$ -spaces

We consider the following diagram:

$$\begin{array}{ccc}
 E & & (F, [\cdot, \cdot]_{F,h}, (\rho, \eta)) \\
 \pi \downarrow & & \downarrow \nu \\
 M & \xrightarrow{h} & N
 \end{array}$$

such that  $(E, \pi, M) = (F, \nu, N)$  and the generalized tangent bundle

$$((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$$

is  $(\rho, \eta)$ -(pseudo)metrizable. Let

$$G = g_{ab} d\tilde{z}^a \otimes d\tilde{z}^a + g_{ab} \delta\tilde{y}^a \otimes \delta\tilde{y}^b$$

be a (pseudo)metrical structure and

$$((\rho, \eta) H, (\rho, \eta) V)$$

a distinguished linear  $(\rho, \eta)$ -connection such that

$$(\rho, \eta) D_X G = 0, \quad \forall X \in \Gamma((\rho, \eta) TE, (\rho, \eta) \tau_E, E).$$

**Definition 6.1** A smooth *Lagrange fundamental function* on the vector bundle  $(E, \pi, M)$  is a mapping  $E \xrightarrow{L} \mathbb{R}$  which satisfies the following conditions:

1.  $L \circ u \in C^\infty(M)$ , for any  $u \in \Gamma(E, \pi, M) \setminus \{0\}$ ;
2.  $L \circ 0 \in C^0(M)$ , where 0 means the null section of  $(E, \pi, M)$ .

If  $(U, s_U)$  is a local vector  $(m+r)$ -chart for  $(E, \pi, M)$ , then real function

$$L_{ab} \stackrel{\text{put}}{=} \frac{\partial^2 L}{\partial y^a \partial y^b} \stackrel{\text{put}}{=} \frac{\partial}{\partial y^a} \left( \frac{\partial}{\partial y^b} (L) \right)$$

is defined on  $\pi^{-1}(U)$ .

**Definition 6.2** If for any local vector  $m+r$ -chart  $(U, s_U)$  of  $(E, \pi, M)$ , we have:

$$(6.2) \quad \text{rank} \|L_{ab}(u_x)\| = r,$$

for any  $u_x \in \pi^{-1}(U) \setminus \{0_x\}$ , then we will say that *the Lagrangian  $L$  is regular*.

**Proposition 6.1** *If the Lagrangian  $L$  is regular, then for any local vector  $m+r$ -chart  $(U, s_U)$  of  $(E, \pi, M)$ , we obtain the real functions  $\tilde{L}^{ab}$  locally defined by*

$$(6.3) \quad \begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\tilde{L}^{ab}} & \mathbb{R} \\ u_x & \longmapsto & \tilde{L}^{ab}(u_x) \end{array},$$

where  $\|\tilde{L}^{ab}(u_x)\| = \|L_{ab}(u_x)\|^{-1}$ , for any  $u_x \in \pi^{-1}(U) \setminus \{0_x\}$ .

**Definition 6.3** A smooth *Finsler fundamental function* on the vector bundle  $(E, \pi, M)$  is a smooth Lagrange fundamental function  $E \xrightarrow{F} \mathbb{R}_+$  which satisfies the following conditions:

1.  $F$  is positively 1-homogenous on the fibres of vector bundle  $(E, \pi, M)$ ;
2. For any local vector  $m+r$ -chart  $(U, s_U)$  of  $(E, \pi, M)$ , the hessian:

$$(6.4) \quad \|F_{ab}^2(u_x)\|$$

is positively define for any  $u_x \in \pi^{-1}(U) \setminus \{0_x\}$ .

**Definition 6.4** If the (pseudo)metrical structure  $G$  is determined by a (pseudo)metrical structure

$$g \in \mathcal{T}_2^0(V(\rho, \eta)TE, (\rho, \eta), \tau_E, E),$$

then the  $(\rho, \eta)$ -(pseudo)metrizable vector bundle

$$((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$$

will be called the *generalized Lagrange  $(\rho, \eta)$ -space*.

In particular, if the (pseudo)metrical structure  $g$  is determined with the help of a Lagrange (Finsler) fundamental function, namely  $g = L_{ab}d\tilde{y}^a \otimes d\tilde{y}^b$  ( $g = F_{ab}^2 d\tilde{y}^a \otimes d\tilde{y}^b$ ), then the  $(\rho, \eta)$ -(pseudo)metrizable vector bundle

$$((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$$

will be called the *Lagrange (Finsler)  $(\rho, \eta)$ -space*.

The generalized Lagrange  $(Id_{TM}, Id_M)$ -spaces, the Lagrange  $(Id_{TM}, Id_M)$ -spaces, and the Finsler  $(Id_{TM}, Id_M)$ -spaces are the usual generalized Lagrange spaces, Lagrange spaces and Finsler spaces.

**Theorem 6.1** *If the (pseudo)metrical structure  $G$  is determined by a (pseudo)metrical structure*

$$g \in \mathcal{T}_2^0(V(\rho, \eta)TE, (\rho, \eta), \tau_E, E),$$

*then, the real local functions:*

$$(6.5) \quad \begin{aligned} (\rho, \eta) H_{bc}^a &= \frac{1}{2} \tilde{g}^{ae} \left( \Gamma(\tilde{\rho}, Id_E) \left( \tilde{\delta}_b \right) g_{ec} + \Gamma(\tilde{\rho}, Id_E) \left( \tilde{\delta}_c \right) g_{be} - \Gamma(\tilde{\rho}, Id_E) \left( \tilde{\delta}_e \right) g_{bc} \right. \\ &\quad \left. - g_{cd} L_{be}^d \circ h \circ \pi + g_{bd} L_{ec}^d \circ h \circ \pi - g_{ed} L_{bc}^d \circ h \circ \pi \right), \\ (\rho, \eta) V_{bc}^a &= \frac{1}{2} \tilde{g}^{ae} \left( \Gamma(\tilde{\rho}, Id_E) \left( \dot{\tilde{\partial}}_c \right) g_{eb} + \Gamma(\tilde{\rho}, Id_E) \left( \dot{\tilde{\partial}}_b \right) g_{ec} - \Gamma(\tilde{\rho}, Id_E) \left( \dot{\tilde{\partial}}_e \right) g_{bc} \right) \end{aligned}$$

*are the components of a normal distinguished linear  $(\rho, \eta)$ -connection with  $(\rho, \eta)$ - $\mathcal{H}$  ( $\mathcal{H}\mathcal{H}$ ) and  $(\rho, \eta)$ - $\mathcal{V}$  ( $\mathcal{V}\mathcal{V}$ ) torsions free such that the generalized tangent bundle  $((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$  becomes generalized Lagrange  $(\rho, \eta)$ -space.*

This normal distinguished linear  $(\rho, \eta)$ -connection will be called *generalized linear  $(\rho, \eta)$ -connection of Levi-Civita type*.

**Corolary 6.1** *In the particular case of Lie algebroids,  $(\eta, h) = (Id_M, Id_M)$ , then we obtain*

$$(6.5)' \quad \begin{aligned} \rho H_{bc}^a &= \frac{1}{2} \tilde{g}^{ae} \left( \Gamma(\tilde{\rho}, Id_E) \left( \tilde{\delta}_b \right) g_{ec} + \Gamma(\tilde{\rho}, Id_E) \left( \tilde{\delta}_c \right) g_{be} - \Gamma(\tilde{\rho}, Id_E) \left( \tilde{\delta}_e \right) g_{bc} \right. \\ &\quad \left. - g_{cd} L_{be}^d \circ \pi + g_{bd} L_{ec}^d \circ \pi - g_{ed} L_{bc}^d \circ \pi \right), \\ \rho V_{bc}^a &= \frac{1}{2} \tilde{g}^{ae} \left( \Gamma(\tilde{\rho}, Id_E) \left( \dot{\tilde{\partial}}_c \right) g_{eb} + \Gamma(\tilde{\rho}, Id_E) \left( \dot{\tilde{\partial}}_b \right) g_{ec} - \Gamma(\tilde{\rho}, Id_E) \left( \dot{\tilde{\partial}}_e \right) g_{bc} \right) \end{aligned}$$

*In the classicale case,  $(\rho, \eta, h) = (Id_{TE}, Id_M, Id_M)$ , then we obtain*

$$(6.5)'' \quad \begin{aligned} H_{bc}^a &= \frac{1}{2} \tilde{g}^{ae} (\delta_b g_{ec} + \delta_c g_{be} - \delta_e g_{bc}) \\ V_{bc}^a &= \frac{1}{2} \tilde{g}^{ae} (\dot{\partial}_c g_{eb} + \dot{\partial}_b g_{ec} - \dot{\partial}_e g_{bc}) \end{aligned}$$

*Moreover, if  $(E, \pi, M) = (TM, \tau_M, M)$ , then we obtain*

$$(6.5)''' \quad \begin{aligned} H_{jk}^i &= \frac{1}{2} \tilde{g}^{ih} (\delta_j g_{hk} + \delta_k g_{jh} - \delta_h g_{jk}) \\ V_{jk}^i &= \frac{1}{2} \tilde{g}^{ih} (\dot{\partial}_k g_{hj} + \dot{\partial}_j g_{hk} - \dot{\partial}_h g_{jk}) \end{aligned}$$

**Theorem 6.2** *Let  $((\rho, \eta)H, (\rho, \eta)V)$  be the normal distinguished linear  $(\rho, \eta)$ -connection presented in the previous theorem.*

*If*

$$\mathbb{T}_{bc}^a \tilde{\delta}_a \otimes d\tilde{z}^b \otimes d\tilde{z}^c \in \mathcal{T}_{20}^{10}((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$$

*and*

$$\mathbb{S}_{bc}^a \dot{\tilde{\partial}}_a \otimes \delta \tilde{y}^b \otimes \delta \tilde{y}^c \in \mathcal{T}_{02}^{01}((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$$

*such that they satisfy the conditions:*

$$\mathbb{T}_{bc}^a = -\mathbb{T}_{cb}^a \wedge \mathbb{S}_{bc}^a = -\mathbb{S}_{cb}^a, \quad \forall b, c \in \overline{1, n},$$

then the following real local functions:

$$(6.6) \quad \begin{aligned} (\rho, \eta) \tilde{H}_{bc}^a &= (\rho, \eta) H_{bc}^a + \frac{1}{2} \tilde{g}^{ae} \left( g_{ed} \mathbb{T}_{bc}^d - g_{bd} \mathbb{T}_{ec}^d + g_{cd} \mathbb{T}_{be}^d \right), \\ (\rho, \eta) \tilde{V}_{bc}^a &= (\rho, \eta) V_{bc}^a + \frac{1}{2} \tilde{g}^{ae} \left( g_{ed} \mathbb{S}_{bc}^d - g_{bd} \mathbb{S}_{ec}^d + g_{cd} \mathbb{S}_{be}^d \right) \end{aligned}$$

are the components of a normal distinguished linear  $(\rho, \eta)$ -connection with  $(\rho, \eta)$ - $\mathcal{H}$  ( $\mathcal{H}\mathcal{H}$ ) and  $(\rho, \eta)$ - $\mathcal{V}$  ( $\mathcal{V}\mathcal{V}$ ) torsions a priori given such that the generalized tangent bundle  $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$  derives generalized Lagrange  $(\rho, \eta)$ -space.

Moreover, we obtain:

$$(6.7) \quad \begin{aligned} \mathbb{T}_{bc}^a &= (\rho, \eta) \tilde{H}_{bc}^a - (\rho, \eta) \tilde{H}_{cb}^a - L_{bc}^a \circ h \circ \pi, \\ \mathbb{S}_{bc}^a &= (\rho, \eta) \tilde{V}_{bc}^a - (\rho, \eta) \tilde{V}_{cb}^a. \end{aligned}$$

**Corollary 6.2** In the particular case of Lie algebroids,  $(\eta, h) = (Id_M, Id_M)$ , then we obtain

$$(6.6)' \quad \begin{aligned} \rho \tilde{H}_{bc}^a &= \rho H_{bc}^a + \frac{1}{2} \tilde{g}^{ae} \left( g_{ed} \mathbb{T}_{bc}^d - g_{bd} \mathbb{T}_{ec}^d + g_{cd} \mathbb{T}_{be}^d \right), \\ \rho \tilde{V}_{bc}^a &= \rho V_{bc}^a + \frac{1}{2} \tilde{g}^{ae} \left( g_{ed} \mathbb{S}_{bc}^d - g_{bd} \mathbb{S}_{ec}^d + g_{cd} \mathbb{S}_{be}^d \right) \end{aligned}$$

and

$$(6.7)' \quad \begin{aligned} \mathbb{T}_{bc}^a &= \rho \tilde{H}_{bc}^a - \rho \tilde{H}_{cb}^a - L_{bc}^a \circ \pi, \\ \mathbb{S}_{bc}^a &= \rho \tilde{V}_{bc}^a - \rho \tilde{V}_{cb}^a. \end{aligned}$$

In the classical case,  $(\rho, \eta, h) = (Id_{TE}, Id_M, Id_M)$ , then we obtain

$$(6.6)'' \quad \begin{aligned} \tilde{H}_{bc}^a &= H_{bc}^a + \frac{1}{2} \tilde{g}^{ae} \left( g_{ed} \mathbb{T}_{bc}^d - g_{bd} \mathbb{T}_{ec}^d + g_{cd} \mathbb{T}_{be}^d \right), \\ \tilde{V}_{bc}^a &= V_{bc}^a + \frac{1}{2} \tilde{g}^{ae} \left( g_{ed} \mathbb{S}_{bc}^d - g_{bd} \mathbb{S}_{ec}^d + g_{cd} \mathbb{S}_{be}^d \right) \end{aligned}$$

and

$$(6.7)'' \quad \begin{aligned} \mathbb{T}_{bc}^a &= \tilde{H}_{bc}^a - \tilde{H}_{cb}^a, \\ \mathbb{S}_{bc}^a &= \tilde{V}_{bc}^a - \tilde{V}_{cb}^a. \end{aligned}$$

In particular, if  $(E, \pi, M) = (TM, \tau_M, M)$ , then we obtain

$$(6.6)''' \quad \begin{aligned} \tilde{H}_{jk}^i &= H_{jk}^i + \frac{1}{2} \tilde{g}^{ie} \left( g_{eh} \mathbb{T}_{jk}^h - g_{jh} \mathbb{T}_{ek}^h + g_{kh} \mathbb{T}_{je}^h \right), \\ \tilde{V}_{jk}^i &= V_{jk}^i + \frac{1}{2} \tilde{g}^{ie} \left( g_{eh} \mathbb{S}_{jk}^h - g_{jh} \mathbb{S}_{ek}^h + g_{kh} \mathbb{S}_{je}^h \right) \end{aligned}$$

and

$$(6.7)''' \quad \begin{aligned} \mathbb{T}_{jk}^i &= \tilde{H}_{jk}^i - \tilde{H}_{kj}^i, \\ \mathbb{S}_{jk}^i &= \tilde{V}_{jk}^i - \tilde{V}_{kj}^i. \end{aligned}$$

## Acknowledgment

I am very grateful to Mrs. Ana-Maria ARCUȘ, who helped me to improve the final english form of the text.

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SECONDARY SCHOOL "CORNELIUS RADU",  
 RADINESTI VILLAGE, 217196, GORJ COUNTY, ROMANIA  
 e-mail: c\_arcus@yahoo.com, c\_arcus@radinesti.ro

# THE METRIZABILITY OF THE GENERALIZED TANGENT BUNDLE OF A VECTOR BUNDLE

by  
CONSTANTIN M. ARCUŞ

## Abstract

A class of metrizable vector bundles in the general framework of generalized Lie algebroids have been presented in the paper [5]. Using a generalized Lie algebroid we obtain the Lie algebroid generalized tangent bundle of a vector bundle. This Lie algebroid is the new example of metrizable vector bundle presented in this paper. A new class of Lagrange spaces, called by use, generalized Lagrange  $(\rho, \eta)$ -space, Lagrange  $(\rho, \eta)$ -space and Finsler  $(\rho, \eta)$ -space are presented. In the particular case of Lie algebroids, new and important results are presented. In particular, if all morphisms are identities morphisms, then the classical results are obtained.

**2000 Mathematics Subject Classification:** 53C05, 53C07, 53C60, 58B20.

**Keywords:** vector bundle, (generalized) Lie algebroid, (linear) connection, natural base, adapted base, (pseudo)metrical structure, distinguished linear connection, metrizable vector bundle.

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# 1 Introduction

The study of the geometry of the usual Lie algebroid

$$((TTM, \tau_{TM}, TM), [\cdot, \cdot]_{TTM}, (Id_{TTM}, Id_{TM}))$$

with a metrical structure

$$g = g_{ij} dy^i \otimes dy^j \in \mathcal{T}_2^0(VTTM, \tau_{TM}, TM),$$

was extensively examined by geometers and physicists in the framework of generalized Lagrange space. The generalized Lagrange spaces were introduced and studied by R. Miron [20]. See also [2, 3, 4] for this topic.

We know that a regular Lagrangian on  $TM$  is a smooth function  $TM \xrightarrow{L} \mathbb{R}$  such that the Hessian matrix with entries

$$g_{ij} = \frac{1}{2} \frac{\partial^2 L}{\partial y^i \partial y^j}$$

is everywhere nondegenerate. If the metrical structure of a generalized Lagrange space is determined by a regular Lagrangian, then we obtain the Lagrange space. (see [22]). The notion of Lagrange space was introduced and studied by J. Kern [16] and R. Miron [18, 19]. The nonlinear connections and the distinguished linear connections which depend only on Lagrangian were presented in the framework of Lagrange space. The geometry of Lagrange spaces have been developed in many proceedings and monographs. [8, 10, 12, 21, 22, 23, 24].

The case when  $L$  is square of a function on  $TM$ , positively, 1-homogeneous with respect to the velocity  $y^i$ , provides an important class of Lagrange spaces called Finsler spaces. The geometry of a Finsler space is a subgeometry of the geometry of the Lie algebroid

$$((TTM, \tau_{TM}, TM), [\cdot, \cdot]_{TTM}, (Id_{TTM}, Id_{TM})).$$

Important contributions to the geometry of Finsler spaces were obtained by M. Abate and G. Patritio [1], D. Bao, S. S. Cern and Z. Shen [9], A. Bejancu [10], L. Berwald [11], H. Busmann [13], E. Cartan [14].

The classical results, were extended to the study of the geometry of the usual Lie algebroid  $((TE, \tau_E, E), [\cdot, \cdot]_{TE}, (Id_{TE}, Id_E))$ , where  $(E, \pi, M)$  is a vector bundle. (see [20, 25, 26])

The generalized Lie algebroid (see [5]) is a new notion necessary to obtain a new class of connections in the Ehresmann sense. Using that, we obtain the Lie algebroid generalized tangent bundle  $\left( ((\rho, \eta)TE, (\rho, \eta)\tau_E, E), [\cdot, \cdot]_{(\rho, \eta)TE}, (\tilde{\rho}, Id_E) \right)$ . In the Sections 2, 3 and 4 we present the basic notions and terminology. (see also [5, 7]) The study of the metrizable of this Lie algebroid is our objective in Section 5. In the particular case of Lie algebroids, we obtain important results. Finally, in Section 6, we introduced a new class of Lagrange spaces, called by use *the generalized Lagrange  $(\rho, \eta)$ -spaces, Lagrange  $(\rho, \eta)$ -spaces and Finsler  $(\rho, \eta)$ -spaces*.

New and important results are obtained in the particular case of Lie algebroids. In particular, if  $(\rho, \eta, h) = (Id_{TM}, Id_M, Id_M)$ , then the classical results are obtained.

## 2 Preliminaries

Let **Vect**, **Liealg**, **Mod**, **Man** and  $\mathbf{B}^{\mathbf{v}}$  be the category of real vector spaces, Lie algebras, modules, manifolds and vector bundles respectively.

We know that if  $(E, \pi, M) \in |\mathbf{B}^{\mathbf{v}}|$ ,  $\Gamma(E, \pi, M) = \{u \in \mathbf{Man}(M, E) : u \circ \pi = Id_M\}$  and  $\mathcal{F}(M) = \mathbf{Man}(M, \mathbb{R})$ , then  $(\Gamma(E, \pi, M), +, \cdot)$  is a  $\mathcal{F}(M)$ -module. If  $(\varphi, \varphi_0) \in \mathbf{B}^{\mathbf{v}}((E, \pi, M), (E', \pi', M'))$  such that  $\varphi_0 \in Iso_{\mathbf{Man}}(M, M')$ , then, using the operation

$$\begin{array}{ccc} \mathcal{F}(M) \times \Gamma(E', \pi', M') & \xrightarrow{\quad \cdot \quad} & \Gamma(E', \pi', M') \\ (f, u') & \longmapsto & f \circ \varphi_0^{-1} \cdot u' \end{array}$$

it results that  $(\Gamma(E', \pi', M'), +, \cdot)$  is a  $\mathcal{F}(M)$ -module and we obtain the **Mod**-morphism

$$\begin{array}{ccc} \Gamma(E, \pi, M) & \xrightarrow{\Gamma(\varphi, \varphi_0)} & \Gamma(E', \pi', M') \\ u & \longmapsto & \Gamma(\varphi, \varphi_0)u \end{array}$$

defined by

$$\Gamma(\varphi, \varphi_0)u(y) = \varphi\left(u_{\varphi_0^{-1}(y)}\right),$$

for any  $y \in M'$ .

Let  $M, N \in |\mathbf{Man}|$ ,  $h \in Iso_{\mathbf{Man}}(M, N)$  and  $\eta \in Iso_{\mathbf{Man}}(N, M)$ .

We know (see [5]) that if  $(F, \nu, N) \in |\mathbf{B}^{\mathbf{v}}|$  so that there exists

$$(\rho, \eta) \in \mathbf{B}^{\mathbf{v}}((F, \nu, N), (TM, \tau_M, M))$$

and also an operation

$$\begin{array}{ccc} \Gamma(F, \nu, N) \times \Gamma(F, \nu, N) & \xrightarrow{[\cdot]_{F,h}} & \Gamma(F, \nu, N) \\ (u, v) & \longmapsto & [u, v]_{F,h} \end{array}$$

with the following properties:

*GLA*<sub>1</sub>. the equality holds good

$$[u, f \cdot v]_{F,h} = f[u, v]_{F,h} + \Gamma(Th \circ \rho, h \circ \eta)(u) f \cdot v,$$

for all  $u, v \in \Gamma(F, \nu, N)$  and  $f \in \mathcal{F}(N)$ .

*GLA*<sub>2</sub>. the 4-tuple  $(\Gamma(F, \nu, N), +, \cdot, [\cdot]_{F,h})$  is a Lie  $\mathcal{F}(N)$ -algebra,

*GLA*<sub>3</sub>. the **Mod**-morphism  $\Gamma(Th \circ \rho, h \circ \eta)$  is a **LieAlg**-morphism of

$$\left(\Gamma(F, \nu, N), +, \cdot, [\cdot]_{F,h}\right)$$

source and

$$(\Gamma(TN, \tau_N, N), +, \cdot, [\cdot]_{TN})$$

target, then the triple  $((F, \nu, N), [\cdot]_{F,h}, (\rho, \eta))$  is called *generalized Lie algebroid*.

In particular, if  $h = Id_M = \eta$ , then we obtain the definition of the Lie algebroid.

Let  $((F, \nu, N), [\cdot]_{F,h}, (\rho, \eta))$  be an generalized Lie algebroid.

- Locally, for any  $\alpha, \beta \in \overline{1, p}$ , we set  $[t_\alpha, t_\beta]_{F, h} = L_{\alpha\beta}^\gamma t_\gamma$ . We easily obtain that  $L_{\alpha\beta}^\gamma = -L_{\beta\alpha}^\gamma$ , for any  $\alpha, \beta, \gamma \in \overline{1, p}$ .

The real local functions  $L_{\alpha\beta}^\gamma$ ,  $\alpha, \beta, \gamma \in \overline{1, p}$  will be called the *structure functions of the generalized Lie algebroid*  $\left((F, \nu, N), [\cdot, \cdot]_{F, h}, (\rho, \eta)\right)$ .

- We assume the following diagrams:

$$\begin{array}{ccccc}
 F & \xrightarrow{\rho} & TM & \xrightarrow{Th} & TN \\
 \downarrow \nu & & \downarrow \tau_M & & \downarrow \tau_N \\
 N & \xrightarrow{\eta} & M & \xrightarrow{h} & N \\
 (\chi^{\tilde{i}}, z^\alpha) & & (x^i, y^i) & & (\chi^{\tilde{i}}, z^{\tilde{i}})
 \end{array}$$

where  $i, \tilde{i} \in \overline{1, m}$  and  $\alpha \in \overline{1, p}$ .

If

$$\begin{aligned}
 (\chi^{\tilde{i}}, z^\alpha) &\longrightarrow (\chi^{\tilde{i}'}(\chi^{\tilde{i}}), z^{\alpha'}(\chi^{\tilde{i}}, z^\alpha)), \\
 (x^i, y^i) &\longrightarrow (x^{\tilde{i}'}(x^i), y^{\tilde{i}'}(x^i, y^i))
 \end{aligned}$$

and

$$(\chi^{\tilde{i}}, z^{\tilde{i}}) \longrightarrow (\chi^{\tilde{i}'}(\chi^{\tilde{i}}), z^{\tilde{i}'}(\chi^{\tilde{i}}, z^{\tilde{i}})),$$

then

$$z^{\alpha'} = \Lambda_\alpha^{\alpha'} z^\alpha,$$

$$y^{\tilde{i}'} = \frac{\partial x^{\tilde{i}'}}{\partial x^i} y^i$$

and

$$z^{\tilde{i}'} = \frac{\partial \chi^{\tilde{i}'}}{\partial \chi^{\tilde{i}}} z^{\tilde{i}}.$$

- We assume that  $(\theta, \mu) \stackrel{put}{=} (Th \circ \rho, h \circ \eta)$ . If  $z^\alpha t_\alpha \in \Gamma(F, \nu, N)$  is arbitrary, then

$$\begin{aligned}
 (2.1) \quad & \Gamma(Th \circ \rho, h \circ \eta)(z^\alpha t_\alpha) f(h \circ \eta(\varkappa)) = \\
 & = \left( \theta_\alpha^{\tilde{i}} z^\alpha \frac{\partial f}{\partial \varkappa^{\tilde{i}}} \right) (h \circ \eta(\varkappa)) = \left( (\rho_\alpha^i \circ h)(z^\alpha \circ h) \frac{\partial f \circ h}{\partial x^i} \right) (\eta(\varkappa)),
 \end{aligned}$$

for any  $f \in \mathcal{F}(N)$  and  $\varkappa \in N$ .

The coefficients  $\rho_\alpha^i$  respectively  $\theta_\alpha^{\tilde{i}}$  change to  $\rho_{\alpha'}^{\tilde{i}'}$  respectively  $\theta_{\alpha'}^{\tilde{i}'}$  according to the rule:

$$(2.2) \quad \rho_{\alpha'}^{\tilde{i}'} = \Lambda_\alpha^{\alpha'} \rho_\alpha^i \frac{\partial x^{\tilde{i}'}}{\partial x^i},$$

respectively

$$(2.3) \quad \theta_{\alpha'}^{\tilde{i}'} = \Lambda_\alpha^{\alpha'} \theta_\alpha^{\tilde{i}} \frac{\partial \varkappa^{\tilde{i}'}}{\partial \varkappa^{\tilde{i}}},$$

where

$$\|\Lambda_{\alpha'}^{\alpha}\| = \|\Lambda_\alpha^{\alpha'}\|^{-1}.$$

*Remark 2.1* The following equalities hold good:

$$(2.4) \quad \rho_\alpha^i \circ h \frac{\partial f \circ h}{\partial x^i} = \left( \theta_\alpha^i \frac{\partial f}{\partial x^i} \right) \circ h, \forall f \in \mathcal{F}(N).$$

and

$$(2.5) \quad \left( L_{\alpha\beta}^\gamma \circ h \right) \left( \rho_\gamma^k \circ h \right) = (\rho_\alpha^i \circ h) \frac{\partial (\rho_\beta^k \circ h)}{\partial x^i} - (\rho_\beta^j \circ h) \frac{\partial (\rho_\alpha^k \circ h)}{\partial x^j}.$$

We have the  $\mathbf{B}^\vee$ -morphism

$$(2.6) \quad \begin{array}{ccc} \pi^*(h^*F) & \hookrightarrow & F \\ \pi^*(h^*\nu) \downarrow & & \downarrow \nu \\ E & \xrightarrow{h \circ \pi} & N \end{array}$$

Let  $\left( \begin{smallmatrix} \pi^*(h^*F) \\ \rho \end{smallmatrix}, Id_E \right)$  be the  $\mathbf{B}^\vee$ -morphism of  $(\pi^*(h^*F), \pi^*(h^*\nu), E)$  source and  $(TE, \tau_E, E)$  target, where

$$(2.7) \quad \begin{array}{ccc} \pi^*(h^*F) & \xrightarrow{\begin{smallmatrix} \pi^*(h^*F) \\ \rho \end{smallmatrix}} & TE \\ Z^\alpha T_\alpha(u_x) & \longmapsto & (Z^\alpha \cdot \rho_\alpha^i \circ h \circ \pi) \frac{\partial}{\partial x^i}(u_x) \end{array}$$

Using the operation

$$\Gamma(\pi^*(h^*F), \pi^*(h^*\nu), E)^2 \xrightarrow{[\cdot]_{\pi^*(h^*F)}} \Gamma(\pi^*(h^*F), \pi^*(h^*\nu), E)$$

defined by

$$(2.8) \quad \begin{aligned} [T_\alpha, T_\beta]_{\pi^*(h^*F)} &= (L_{\alpha\beta}^\gamma \circ h \circ \pi) T_\gamma, \\ [T_\alpha, fT_\beta]_{\pi^*(h^*F)} &= f (L_{\alpha\beta}^\gamma \circ h \circ \pi) T_\gamma + (\rho_\alpha^i \circ h \circ \pi) \frac{\partial f}{\partial x^i} T_\beta, \\ [fT_\alpha, T_\beta]_{\pi^*(h^*F)} &= -[T_\beta, fT_\alpha]_{\pi^*(h^*F)}, \end{aligned}$$

for any  $f \in \mathcal{F}(E)$ , it results that

$$\left( (\pi^*(h^*F), \pi^*(h^*\nu), E), [\cdot, \cdot]_{\pi^*(h^*F)}, \left( \begin{smallmatrix} \pi^*(h^*F) \\ \rho \end{smallmatrix}, Id_E \right) \right)$$

is a Lie algebroid.

### 3 Natural and adapted basis

We consider the following diagram:

$$(3.1) \quad \begin{array}{ccc} E & & (F, [\cdot, \cdot]_{F,h}, (\rho, \eta)) \\ \pi \downarrow & & \downarrow \nu \\ M & \xrightarrow{h} & N \end{array}$$

where  $(E, \pi, M)$  is a vector bundle and  $\left((F, \nu, N), [\cdot, \cdot]_{F,h}, (\rho, \eta)\right)$  is a generalized Lie algebroid.

We take  $(x^i, y^a)$  as canonical local coordinates on  $(E, \pi, M)$ , where  $i \in \overline{1, m}$  and  $a \in \overline{1, r}$ . Let

$$(x^i, y^a) \longrightarrow (x^{\check{i}}(x^i), y^{\check{a}}(x^i, y^a))$$

be a change of coordinates on  $(E, \pi, M)$ . Then the coordinates  $y^a$  change to  $y^{\check{a}}$  by the rule:

$$(3.2) \quad y^{\check{a}} = M_a^{\check{a}} y^a.$$

Let

$$(3.3) \quad \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^a}\right) \stackrel{put}{=} (\partial_i, \dot{\partial}_a)$$

be the natural base of the Lie algebroid  $((TE, \tau_E, E), [\cdot, \cdot]_{TE}, (Id_{TE}, Id_E))$ .

For any sections

$$Z^\alpha T_\alpha \in \Gamma(\pi^*(h^*F), \pi^*(h^*F), E)$$

and

$$Y^a \dot{\partial}_a \in \Gamma(VTE, \tau_E, E)$$

we obtain the section

$$\begin{aligned} Z^\alpha \tilde{\partial}_\alpha + Y^a \dot{\partial}_a &=: Z^\alpha (T_\alpha \oplus (\rho_\alpha^i \circ h \circ \pi) \partial_i) + Y^a (0_{\pi^*(h^*F)} \oplus \dot{\partial}_a) \\ &= Z^\alpha T_\alpha \oplus (Z^\alpha (\rho_\alpha^i \circ h \circ \pi) \partial_i + Y^a \dot{\partial}_a) \in \Gamma(\pi^*(h^*F) \oplus TE, \overset{\oplus}{\pi}, E). \end{aligned}$$

Since we have

$$\begin{aligned} Z^\alpha \tilde{\partial}_\alpha + Y^a \dot{\partial}_a &= 0 \\ \Updownarrow \\ Z^\alpha T_\alpha &= 0 \wedge Z^\alpha (\rho_\alpha^i \circ h \circ \pi) \partial_i + Y^a \dot{\partial}_a = 0, \end{aligned}$$

it implies  $Z^\alpha = 0$ ,  $\alpha \in \overline{1, p}$  and  $Y^a = 0$ ,  $a \in \overline{1, r}$ .

Therefore, the sections  $\tilde{\partial}_1, \dots, \tilde{\partial}_p, \dot{\partial}_1, \dots, \dot{\partial}_r$  are linearly independent.

We consider the vector subbundle  $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$  of the vector bundle  $(\pi^*(h^*F) \oplus TE, \overset{\oplus}{\pi}, E)$ , for which the  $\mathcal{F}(E)$ -module of sections is the  $\mathcal{F}(E)$ -submodule of  $(\Gamma(\pi^*(h^*F) \oplus TE, \overset{\oplus}{\pi}, E), +, \cdot)$ , generated by the set of sections  $(\tilde{\partial}_\alpha, \dot{\partial}_a)$  which is called the *natural*  $(\rho, \eta)$ -base.

The matrix of coordinate transformation on  $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$  at a change of fibred charts is

$$(3.4) \quad \left\| \begin{array}{cc} \Lambda_\alpha^{\check{\alpha}} \circ h \circ \pi & 0 \\ (\rho_\alpha^i \circ h \circ \pi) \frac{\partial M_b^{\check{a}} \circ \pi}{\partial x_i} y^b & M_a^{\check{a}} \circ \pi \end{array} \right\|.$$

We have the following

**Theorem 3.1** Let  $(\tilde{\rho}, Id_E)$  be the  $\mathbf{B}^v$ -morphism of  $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$  source and  $(TE, \tau_E, E)$  target, where

$$(3.5) \quad \begin{aligned} & (\rho, \eta) TE \xrightarrow{\tilde{\rho}} TE \\ & \left( Z^\alpha \tilde{\partial}_\alpha + Y^a \dot{\tilde{\partial}}_a \right) (u_x) \mapsto \left( Z^\alpha (\rho_\alpha^i \circ h \circ \pi) \partial_i + Y^a \dot{\partial}_a \right) (u_x) \end{aligned}$$

Using the operation

$$\Gamma((\rho, \eta) TE, (\rho, \eta) \tau_E, E)^2 \xrightarrow{[\cdot]_{(\rho, \eta) TE}} \Gamma((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$$

defined by

$$(3.6) \quad \begin{aligned} & \left[ \left( Z_1^\alpha \tilde{\partial}_\alpha + Y_1^a \dot{\tilde{\partial}}_a \right), \left( Z_2^\beta \tilde{\partial}_\beta + Y_2^b \dot{\tilde{\partial}}_b \right) \right]_{(\rho, \eta) TE} \\ &= \left[ Z_1^\alpha T_a, Z_2^\beta T_\beta \right]_{\pi^*(h^*F)} \oplus \left[ (\rho_\alpha^i \circ h \circ \pi) Z_1^\alpha \partial_i + Y_1^a \dot{\partial}_a, \right. \\ & \quad \left. (\rho_\beta^j \circ h \circ \pi) Z_2^\beta \partial_j + Y_2^b \dot{\partial}_b \right]_{TE}, \end{aligned}$$

for any  $\left( Z_1^\alpha \tilde{\partial}_\alpha + Y_1^a \dot{\tilde{\partial}}_a \right)$  and  $\left( Z_2^\beta \tilde{\partial}_\beta + Y_2^b \dot{\tilde{\partial}}_b \right)$ , we obtain that the couple

$$([\cdot]_{(\rho, \eta) TE}, (\tilde{\rho}, Id_E))$$

is a Lie algebroid structure for the vector bundle  $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$ .

The Lie algebroid

$$\left( ((\rho, \eta) TE, (\rho, \eta) \tau_E, E), [\cdot]_{(\rho, \eta) TE}, (\tilde{\rho}, Id_E) \right),$$

is called the *Lie algebroid generalized tangent bundle*. (see [5, 7])

We consider the  $\mathbf{B}^v$ -morphism  $((\rho, \eta) \pi!, Id_E)$  given by the commutative diagram

$$(3.7) \quad \begin{array}{ccc} (\rho, \eta) TE & \xrightarrow{(\rho, \eta) \pi!} & \pi^*(h^*F) \\ (\rho, \eta) \tau_E \downarrow & & \downarrow pr_1 \\ E & \xrightarrow{id_E} & E \end{array}$$

This is defined as:

$$(3.8) \quad (\rho, \eta) \pi! \left( \left( Z^\alpha \tilde{\partial}_\alpha + Y^a \dot{\tilde{\partial}}_a \right) (u_x) \right) = (Z^\alpha T_\alpha) (u_x),$$

for any  $\left( Z^\alpha \tilde{\partial}_\alpha + Y^a \dot{\tilde{\partial}}_a \right) \in \Gamma((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$ .

Using the  $\mathbf{B}^v$ -morphisms (2.6) and (3.7) we obtain the *tangent*  $(\rho, \eta)$ -application  $((\rho, \eta) T\pi, h \circ \pi)$  of  $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$  source and  $(F, \nu, N)$  target.

**Definition 3.1** The kernel of the tangent  $(\rho, \eta)$ -application is written

$$(V(\rho, \eta) TE, (\rho, \eta) \tau_E, E)$$



and it is called *the vertical subbundle*.

We remark that the set  $\left\{ \dot{\tilde{\partial}}_a, a \in \overline{1, r} \right\}$  is a base of the  $\mathcal{F}(E)$ -module

$$(\Gamma(V(\rho, \eta)TE, (\rho, \eta)\tau_E, E), +, \cdot).$$

**Proposition 3.1** *The short sequence of vector bundles*

$$(3.9) \quad \begin{array}{ccccccccc} 0 & \xrightarrow{i} & V(\rho, \eta)TE & \xrightarrow{i} & (\rho, \eta)TE & \xrightarrow{(\rho, \eta)\pi^!} & \pi^*(h^*F) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ E & \xrightarrow{Id_E} & E & \xrightarrow{Id_E} & E & \xrightarrow{Id_E} & E & \xrightarrow{Id_E} & E \end{array}$$

is exact.

Let  $(\rho, \eta)\Gamma$  be a  $(\rho, \eta)$ -connection for the vector bundle  $(E, \pi, M)$ , i. e. a **Man**-morphism  $(\rho, \eta)\Gamma$  of  $(\rho, \eta)TE$  source and  $V(\rho, \eta)TE$  target defined by

$$(3.10) \quad (\rho, \eta)\Gamma \left( Z^\alpha \tilde{\partial}_\alpha + Y^a \dot{\tilde{\partial}}_a \right) (u_x) = (Y^a + (\rho, \eta)\Gamma_\alpha^a Z^\alpha) \dot{\tilde{\partial}}_a (u_x),$$

so that the **B<sup>V</sup>**-morphism  $((\rho, \eta)\Gamma, Id_E)$  is a split to the left in the previous exact sequence. Its components satisfy the law of transformation

$$(3.11) \quad (\rho, \eta)\Gamma_\gamma^{a'} = M_a^{a'} \circ \pi \left[ \rho_\gamma^i \circ h \circ \pi \frac{\partial M_k^a \circ \pi}{\partial x^i} y^b + (\rho, \eta)\Gamma_\gamma^a \right] \Lambda_\gamma^{\gamma'} \circ h \circ \pi.$$

In the particular case of Lie algebroids,  $(\eta, h) = (Id_M, Id_M)$ , we obtain

$$(3.11)' \quad \rho \Gamma_\gamma^{a'} = M_a^{a'} \circ \pi \left[ \rho_\gamma^i \circ \pi \frac{\partial M_k^a \circ \pi}{\partial x^i} y^b + \rho \Gamma_\gamma^a \right] \Lambda_\gamma^{\gamma'} \circ \pi.$$

In the classical case,  $(\rho, \eta, h) = (Id_{TE}, Id_M, Id_M)$ , we obtain

$$(3.11)'' \quad \Gamma_k^{a'} = M_a^{a'} \circ \pi \left[ \frac{\partial M_k^a \circ \pi}{\partial x^i} y^b + \Gamma_k^a \right] \frac{\partial x^k}{\partial x^{k'}} \circ \pi.$$

The kernel of the **B<sup>V</sup>**-morphism  $((\rho, \eta)\Gamma, Id_E)$  is written  $(H(\rho, \eta)TE, (\rho, \eta)\tau_E, E)$  and is called the *horizontal vector subbundle*. (see [5, 7])

We put the problem of finding a base for the  $\mathcal{F}(E)$ -module

$$(\Gamma(H(\rho, \eta)TE, (\rho, \eta)\tau_E, E), +, \cdot)$$

of the type

$$\tilde{\delta}_\alpha = Z_\alpha^\beta \tilde{\partial}_\beta + Y_\alpha^a \dot{\tilde{\partial}}_a, \alpha \in \overline{1, r}$$

which satisfies the following conditions:

$$(3.12) \quad \begin{aligned} \Gamma((\rho, \eta)\pi^!, Id_E) \left( \tilde{\delta}_\alpha \right) &= T_\alpha, \\ \Gamma((\rho, \eta)\Gamma, Id_E) \left( \tilde{\delta}_\alpha \right) &= 0. \end{aligned}$$

Then we obtain the sections

$$(3.13) \quad \frac{\delta}{\delta \tilde{z}^\alpha} = \tilde{\partial}_\alpha - (\rho, \eta)\Gamma_\alpha^a \dot{\tilde{\partial}}_a = T_\alpha \oplus \left( (\rho_\alpha^i \circ h \circ \pi) \partial_i - (\rho, \eta)\Gamma_\alpha^a \dot{\partial}_a \right).$$

such that their law of change is a tensorial law under a change of vector fiber charts.

The base  $\left(\tilde{\delta}_\alpha, \dot{\tilde{\partial}}_a\right)$  will be called the *adapted*  $(\rho, \eta)$ -base.

*Remark 3.2* The following equality holds good

$$(3.14) \quad \Gamma(\tilde{\rho}, Id_E) \left(\tilde{\delta}_\alpha\right) = (\rho_\alpha^i \circ h \circ \pi) \partial_i - (\rho, \eta) \Gamma_\alpha^a \dot{\partial}_a.$$

Moreover, if  $(\rho, \eta) \Gamma$  is the  $(\rho, \eta)$ -connection associated to a connection  $\Gamma$  (see [5]), then we obtain

$$(3.15) \quad \Gamma(\tilde{\rho}, Id_E) \left(\tilde{\delta}_\alpha\right) = (\rho_\alpha^i \circ h \circ \pi) \delta_i,$$

where  $\left(\delta_i, \dot{\partial}_a\right)$  is the adapted base for the  $\mathcal{F}(E)$ -module  $(\Gamma(TE, \tau_E, E), +, \cdot)$ .

Let  $(d\tilde{z}^\alpha, d\tilde{y}^b)$  be the natural dual  $(\rho, \eta)$ -base of natural  $(\rho, \eta)$ -base  $\left(\tilde{\delta}_\alpha, \dot{\tilde{\partial}}_a\right)$ .

This is determined by the equations

$$\begin{cases} \langle d\tilde{z}^\alpha, \tilde{\partial}_\beta \rangle = \delta_\beta^\alpha, & \langle d\tilde{z}^\alpha, \dot{\tilde{\partial}}_a \rangle = 0, \\ \langle d\tilde{y}^a, \tilde{\partial}_\beta \rangle = 0, & \langle d\tilde{y}^a, \dot{\tilde{\partial}}_b \rangle = \delta_b^a. \end{cases}$$

We consider the problem of finding a base for the  $\mathcal{F}(E)$ -module

$$(\Gamma((V(\rho, \eta)TE)^*, ((\rho, \eta)\tau_E)^*, E), +, \cdot)$$

of the type

$$\delta\tilde{y}^a = \theta_\alpha^a d\tilde{z}^\alpha + \omega_b^a d\tilde{y}^b, \quad a \in \overline{1, n}$$

which satisfies the following conditions:

$$(3.16) \quad \left\langle \delta\tilde{y}^a, \dot{\tilde{\partial}}_a \right\rangle = 1 \wedge \left\langle \delta\tilde{y}^a, \tilde{\delta}_\alpha \right\rangle = 0.$$

We obtain the sections

$$(3.17) \quad \delta\tilde{y}^a = (\rho, \eta) \Gamma_\alpha^a d\tilde{z}^\alpha + d\tilde{y}^a, \quad a \in \overline{1, n}.$$

such that their changing rule is tensorial under a change of vector fiber charts. The base  $(d\tilde{z}^\alpha, \delta\tilde{y}^a)$  will be called the *adapted dual*  $(\rho, \eta)$ -base.

## 4 Tensor $d$ -fields. Distinguished linear $(\rho, \eta)$ -connections

We consider the following diagram:

$$\begin{array}{ccc} E & & (F, [\cdot, \cdot]_{F,h}, (\rho, \eta)) \\ \pi \downarrow & & \downarrow \nu \\ M & \xrightarrow{h} & N \end{array}$$

where  $(E, \pi, M) \in |\mathbf{B}^V|$  and  $\left( (F, \nu, N), [\cdot, \cdot]_{F,h}, (\rho, \eta) \right)$  is a generalized Lie algebroid.

Let

$$(\mathcal{T}_{q,s}^{p,r}((\rho, \eta) TE, (\rho, \eta) \tau_E, E), +, \cdot)$$

be the  $\mathcal{F}(E)$ -module of tensor fields by  $(\frac{p,r}{q,s})$ -type from the generalized tangent bundle

$$(H(\rho, \eta) TE, (\rho, \eta) \tau_E, E) \oplus (V(\rho, \eta) TE, (\rho, \eta) \tau_E, E).$$

An arbitrarily tensor field  $T$  is written as

$$T = T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \tilde{\delta}_{\alpha_1} \otimes \dots \otimes \tilde{\delta}_{\alpha_p} \otimes d\tilde{z}^{\beta_1} \otimes \dots \otimes d\tilde{z}^{\beta_q} \otimes \tilde{\partial}_{a_1} \otimes \dots \otimes \tilde{\partial}_{a_r} \otimes \delta \tilde{y}^{b_1} \otimes \dots \otimes \delta \tilde{y}^{b_s}.$$

Let

$$(\mathcal{T}((\rho, \eta) TE, (\rho, \eta) \tau_E, E), +, \cdot, \otimes)$$

be the tensor fields algebra of generalized tangent bundle  $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$ .

If  $T_1 \in \mathcal{T}_{q_1, s_1}^{p_1, r_1}((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$  and  $T_2 \in \mathcal{T}_{q_2, s_2}^{p_2, r_2}((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$ , then the components of product tensor field  $T_1 \otimes T_2$  are the products of local components of  $T_1$  and  $T_2$ . Therefore, we obtain  $T_1 \otimes T_2 \in \mathcal{T}_{q_1+q_2, s_1+s_2}^{p_1+p_2, r_1+r_2}((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$ .

Let  $\mathcal{DT}((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$  be the family of tensor fields

$$T \in \mathcal{T}((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$$

for which there exists

$$T_1 \in \mathcal{T}_{q,0}^{p,0}((\rho, \eta) TE, (\rho, \eta) \tau_E, E) \text{ and } T_2 \in \mathcal{T}_{0,s}^{0,r}((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$$

such that  $T = T_1 + T_2$ .

The  $\mathcal{F}(E)$ -module  $(\mathcal{DT}((\rho, \eta) TE, (\rho, \eta) \tau_E, E), +, \cdot)$  will be called the *module of distinguished tensor fields* or the *module of tensor d-fields*.

*Remark 5.1* The elements of

$$\Gamma((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$$

respectively

$$\Gamma(((\rho, \eta) TE)^*, ((\rho, \eta) \tau_E)^*, E)$$

are tensor  $d$ -fields.

**Definition 4.1** Let  $(E, \pi, M)$  be a vector bundle endowed with a  $(\rho, \eta)$ -connection  $(\rho, \eta) \Gamma$  and let

$$(4.1) \quad (X, T) \xrightarrow{(\rho, \eta) D} (\rho, \eta) D_X T$$

be a covariant  $(\rho, \eta)$ -derivative for the tensor algebra of the generalized tangent bundle

$$((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$$

which preserves the horizontal and vertical *IDS* by parallelism. (see [6])

The real local functions

$$((\rho, \eta) H_{\beta\gamma}^\alpha, (\rho, \eta) H_{b\gamma}^a, (\rho, \eta) V_{\beta c}^\alpha, (\rho, \eta) V_{bc}^a)$$

defined by the following equalities:

$$(4.2) \quad \begin{aligned} (\rho, \eta) D_{\tilde{\delta}_\gamma} \tilde{\delta}_\beta &= (\rho, \eta) H_{\beta\gamma}^\alpha \tilde{\delta}_\alpha, & (\rho, \eta) D_{\tilde{\delta}_\gamma} \tilde{\partial}_b &= (\rho, \eta) H_{b\gamma}^a \tilde{\partial}_a \\ (\rho, \eta) D_{\tilde{\partial}_c} \tilde{\delta}_\beta &= (\rho, \eta) V_{\beta c}^\alpha \tilde{\delta}_\alpha, & (\rho, \eta) D_{\tilde{\partial}_c} \tilde{\partial}_b &= (\rho, \eta) V_{bc}^a \tilde{\partial}_a \end{aligned}$$

are the components of a linear  $(\rho, \eta)$ -connection  $((\rho, \eta) H, (\rho, \eta) V)$  for the generalized tangent bundle  $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$  which will be called the *distinguished linear  $(\rho, \eta)$ -connection*.

If  $h = Id_M$ , then the distinguished linear  $(Id_{TM}, Id_M)$ -connection is the classical *distinguished linear connection*.

The components of a distinguished linear connection  $(H, V)$  will be denoted

$$(H_{jk}^i, H_{bk}^a, V_{jc}^i, V_{bc}^a).$$

**Theorem 4.1** *If  $((\rho, \eta)H, (\rho, \eta)V)$  is a distinguished linear  $(\rho, \eta)$ -connection for the generalized tangent bundle  $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$ , then its components satisfy the change relations:*

$$(4.3) \quad \begin{aligned} (\rho, \eta) H_{\beta\gamma'}^\alpha &= \Lambda_\alpha^\alpha \circ h \circ \pi \cdot \left[ \Gamma(\tilde{\rho}, Id_E) \left( \tilde{\delta}_\gamma \right) \left( \Lambda_\beta^\alpha \circ h \circ \pi \right) + \right. \\ &\quad \left. + (\rho, \eta) H_{\beta\gamma}^\alpha \cdot \Lambda_\beta^\beta \circ h \circ \pi \right] \cdot \Lambda_{\gamma'}^\gamma \circ h \circ \pi, \\ (\rho, \eta) H_{b\gamma'}^a &= M_a^a \circ \pi \cdot \left[ \Gamma(\tilde{\rho}, Id_E) \left( \tilde{\delta}_\gamma \right) (M_b^a \circ \pi) + \right. \\ &\quad \left. + (\rho, \eta) H_{b\gamma}^a \cdot M_b^b \circ \pi \right] \cdot \Lambda_{\gamma'}^\gamma \circ h \circ \pi, \\ (\rho, \eta) V_{\beta c'}^\alpha &= \Lambda_\alpha^\alpha \circ h \circ \pi \cdot (\rho, \eta) V_{\beta c}^\alpha \cdot \Lambda_\beta^\beta \circ h \circ \pi \cdot M_c^c \circ \pi, \\ (\rho, \eta) V_{b c'}^a &= M_a^a \circ \pi \cdot (\rho, \eta) V_{bc}^a \cdot M_b^b \circ \pi \cdot M_c^c \circ \pi. \end{aligned}$$

**Corollary 4.1** *In the particular case of Lie algebroids,  $(\eta, h) = (Id_M, Id_M)$ , we obtain*

$$(4.3)' \quad \begin{aligned} \rho H_{\beta\gamma'}^\alpha &= \Lambda_\alpha^\alpha \circ \pi \cdot \left[ \Gamma(\tilde{\rho}, Id_E) \left( \tilde{\delta}_\gamma \right) \left( \Lambda_\beta^\alpha \circ \pi \right) + \rho H_{\beta\gamma}^\alpha \cdot \Lambda_\beta^\beta \circ \pi \right] \cdot \Lambda_{\gamma'}^\gamma \circ \pi \\ \rho H_{b\gamma'}^a &= M_a^a \circ \pi \cdot \left[ \Gamma(\tilde{\rho}, Id_E) \left( \tilde{\delta}_\gamma \right) (M_b^a \circ \pi) + \rho H_{b\gamma}^a \cdot M_b^b \circ \pi \right] \cdot \Lambda_{\gamma'}^\gamma \circ \pi, \\ \rho V_{\beta c'}^\alpha &= \Lambda_\alpha^\alpha \circ \pi \cdot \rho V_{\beta c}^\alpha \cdot \Lambda_\beta^\beta \circ \pi \cdot M_c^c \circ \pi, \\ \rho V_{b c'}^a &= M_a^a \circ \pi \cdot \rho V_{bc}^a \cdot M_b^b \circ \pi \cdot M_c^c \circ \pi. \end{aligned}$$

*In the classical case,  $(\rho, \eta, h) = (Id_{TE}, Id_M, Id_M)$ , we obtain that the components of a distinguished linear connection  $(H, V)$  verify the change relations:*

$$(4.3)'' \quad \begin{aligned} H_{jk'}^i &= \frac{\partial x^i}{\partial x^k} \circ \pi \cdot \left[ \frac{\delta}{\delta x^k} \left( \frac{\partial x^i}{\partial x^j} \circ \pi \right) + H_{jk}^i \cdot \frac{\partial x^j}{\partial x^k} \circ \pi \right] \cdot \frac{\partial x^k}{\partial x^k} \circ \pi, \\ H_{bk'}^a &= M_a^a \circ \pi \cdot \left[ \frac{\delta}{\delta x^k} (M_b^a \circ \pi) + H_{bk}^a \cdot M_b^b \circ \pi \right] \cdot \frac{\partial x^k}{\partial x^k} \circ \pi, \\ V_{jc'}^i &= \frac{\partial x^i}{\partial x^j} \circ \pi \cdot V_{jc}^i \cdot \frac{\partial x^j}{\partial x^j} \circ \pi \cdot M_c^c \circ \pi, \\ V_{b c'}^a &= M_a^a \circ \pi \cdot V_{bc}^a \cdot M_b^b \circ \pi \cdot M_c^c \circ \pi. \end{aligned}$$

**Example 4.1** If  $(E, \pi, M)$  is a vector bundle endowed with the  $(\rho, \eta)$ -connection  $(\rho, \eta)\Gamma$ , then the local real functions

$$(4.4) \quad \left( \frac{\partial(\rho, \eta)\Gamma_\gamma^a}{\partial y^b}, \frac{\partial(\rho, \eta)\Gamma_\gamma^a}{\partial y^b}, 0, 0 \right)$$

are the components of a distinguished linear  $(\rho, \eta)$ -connection for  $((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$ , which will be called the *Berwald linear  $(\rho, \eta)$ -connection*.

The Berwald linear  $(Id_{TM}, Id_M)$ -connection are the usual *Berwald linear connection*.

**Theorem 4.2** *If the generalized tangent bundle  $((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$  is endowed with a distinguished linear  $(\rho, \eta)$ -connection  $((\rho, \eta)H, (\rho, \eta)V)$ , then for any*

$$X = Z^\alpha \tilde{\delta}_\alpha + Y^a \dot{\tilde{\partial}}_a \in \Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$$

and for any

$$T \in \mathcal{T}_{qs}^{pr}((\rho, \eta)TE, (\rho, \eta)\tau_E, E),$$

we obtain the formula:

$$(4.5) \quad \begin{aligned} & (\rho, \eta) D_X \left( T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \tilde{\delta}_{\alpha_1} \otimes \dots \otimes \tilde{\delta}_{\alpha_p} \otimes d\tilde{z}^{\beta_1} \otimes \dots \otimes \right. \\ & \quad \left. \otimes d\tilde{z}^{\beta_q} \otimes \dot{\tilde{\partial}}_{a_1} \otimes \dots \otimes \dot{\tilde{\partial}}_{a_r} \otimes \delta \tilde{y}^{b_1} \otimes \dots \otimes \delta \tilde{y}^{b_s} \right) = \\ & = Z^\gamma T_{\beta_1 \dots \beta_q b_1 \dots b_s | \gamma}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \tilde{\delta}_{\alpha_1} \otimes \dots \otimes \tilde{\delta}_{\alpha_p} \otimes d\tilde{z}^{\beta_1} \otimes \dots \otimes d\tilde{z}^{\beta_q} \otimes \dot{\tilde{\partial}}_{a_1} \otimes \dots \otimes \\ & \quad \otimes \dot{\tilde{\partial}}_{a_r} \otimes \delta \tilde{y}^{b_1} \otimes \dots \otimes \delta \tilde{y}^{b_s} + Y^c T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} |_c \tilde{\delta}_{\alpha_1} \otimes \dots \otimes \\ & \quad \otimes \tilde{\delta}_{\alpha_p} \otimes d\tilde{z}^{\beta_1} \otimes \dots \otimes d\tilde{z}^{\beta_q} \otimes \dot{\tilde{\partial}}_{a_1} \otimes \dots \otimes \dot{\tilde{\partial}}_{a_r} \otimes \delta \tilde{y}^{b_1} \otimes \dots \otimes \delta \tilde{y}^{b_s}, \end{aligned}$$

where

$$(4.6) \quad \begin{aligned} & T_{\beta_1 \dots \beta_q b_1 \dots b_s | \gamma}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} = \Gamma(\tilde{\rho}, Id_E) \left( \tilde{\delta}_\gamma \right) T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \\ & + (\rho, \eta) H_{\alpha \gamma}^{\alpha_1} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_2 \dots \alpha_p a_1 \dots a_r} + \dots + (\rho, \eta) H_{\alpha \gamma}^{\alpha_p} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_{p-1} a_1 \dots a_r} \\ & - (\rho, \eta) H_{\beta_1 \gamma}^\beta T_{\beta \beta_2 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} - \dots - (\rho, \eta) H_{\beta_q \gamma}^\beta T_{\beta_1 \dots \beta_{q-1} \beta b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \\ & + (\rho, \eta) H_{a \gamma}^{a_1} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_2 \dots a_r} + \dots + (\rho, \eta) H_{a \gamma}^{a_r} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_{r-1} a} \\ & - (\rho, \eta) H_{b_1 \gamma}^b T_{\beta_1 \dots \beta_q b b_2 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} - \dots - (\rho, \eta) H_{b_s \gamma}^b T_{\beta_1 \dots \beta_q b_1 \dots b_{s-1} b}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \end{aligned}$$

and

$$(4.7) \quad \begin{aligned} & T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} |_c = \Gamma(\tilde{\rho}, Id_E) \left( \dot{\tilde{\partial}}_c \right) T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \\ & + (\rho, \eta) V_{\alpha c}^{\alpha_1} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_2 \dots \alpha_p a_1 \dots a_r} + \dots + (\rho, \eta) V_{\alpha c}^{\alpha_p} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_{p-1} a_1 \dots a_r} \\ & - (\rho, \eta) V_{\beta_1 c}^\beta T_{\beta \beta_2 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} - \dots - (\rho, \eta) V_{\beta_q c}^\beta T_{\beta_1 \dots \beta_{q-1} \beta b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \\ & + (\rho, \eta) V_{a c}^{a_1} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_2 \dots a_r} + \dots + (\rho, \eta) V_{a c}^{a_r} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_{r-1} a} \\ & - (\rho, \eta) V_{b_1 c}^b T_{\beta_1 \dots \beta_q b b_2 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} - \dots - (\rho, \eta) V_{b_s c}^b T_{\beta_1 \dots \beta_q b_1 \dots b_{s-1} b}^{\alpha_1 \dots \alpha_p a_1 \dots a_r}. \end{aligned}$$

**Corollary 4.2** *In the particular case of Lie algebroids,  $(\eta, h) = (Id_M, Id_M)$ , we obtain*

$$\begin{aligned}
(4.6)' \quad T_{\beta_1 \dots \beta_q b_1 \dots b_s | \gamma}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} &= \Gamma(\tilde{\rho}, Id_E) \left( \tilde{\delta}_\gamma \right) T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \\
&+ \rho H_{\alpha \gamma}^{\alpha_1} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_2 \dots \alpha_p a_1 \dots a_r} + \dots + \rho H_{\alpha \gamma}^{\alpha_p} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_{p-1} a_1 \dots a_r} \\
&- \rho H_{\beta_1 \gamma}^\beta T_{\beta_2 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} - \dots - \rho H_{\beta_q \gamma}^\beta T_{\beta_1 \dots \beta_{q-1} b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \\
&+ \rho H_{a \gamma}^{a_1} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_2 \dots a_r} + \dots + \rho H_{a \gamma}^{a_r} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_{r-1} a} \\
&- \rho H_{b_1 \gamma}^b T_{\beta_1 \dots \beta_q b b_2 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} - \dots - \rho H_{b_s \gamma}^b T_{\beta_1 \dots \beta_q b_1 \dots b_{s-1} b}^{\alpha_1 \dots \alpha_p a_1 \dots a_r}
\end{aligned}$$

and

$$\begin{aligned}
(4.7)' \quad T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} |_{c=} &= \Gamma(\tilde{\rho}, Id_E) \left( \dot{\partial}_c \right) T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \\
&+ \rho V_{\alpha c}^{\alpha_1} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_2 \dots \alpha_p a_1 \dots a_r} + \dots + \rho V_{\alpha c}^{\alpha_p} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_{p-1} a_1 \dots a_r} \\
&- \rho V_{\beta_1 c}^\beta T_{\beta_2 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} - \dots - \rho V_{\beta_q c}^\beta T_{\beta_1 \dots \beta_{q-1} b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \\
&+ \rho V_{a c}^{a_1} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_2 \dots a_r} + \dots + \rho V_{a c}^{a_r} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_{r-1} a} \\
&- \rho V_{b_1 c}^b T_{\beta_1 \dots \beta_q b b_2 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} - \dots - \rho V_{b_s c}^b T_{\beta_1 \dots \beta_q b_1 \dots b_{s-1} b}^{\alpha_1 \dots \alpha_p a_1 \dots a_r}.
\end{aligned}$$

In the classical case,  $(\rho, \eta, h) = (Id_{TE}, Id_M, Id_M)$ , we obtain

$$\begin{aligned}
(4.6)'' \quad T_{j_1 \dots j_q b_1 \dots b_s | k}^{i_1 \dots i_p a_1 \dots a_r} &= \delta_k \left( T_{j_1 \dots j_q b_1 \dots b_s}^{i_1 \dots i_p a_1 \dots a_r} \right) \\
&+ H_{ik}^{i_1} T_{j_1 \dots j_q b_1 \dots b_s}^{i_2 \dots i_p a_1 \dots a_r} + \dots + H_{ik}^{i_p} T_{j_1 \dots j_q b_1 \dots b_s}^{i_1 \dots i_{p-1} a_1 \dots a_r} \\
&- H_{j_1 k}^{j_1} T_{j_2 \dots j_q b_1 \dots b_s}^{i_1 \dots i_p a_1 \dots a_r} - \dots - H_{j_q k}^{j_q} T_{j_1 \dots j_{q-1} b_1 \dots b_s}^{i_1 \dots i_p a_1 \dots a_r} \\
&+ H_{ak}^{a_1} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_2 \dots a_r} + \dots + H_{ak}^{a_r} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_{r-1} a} \\
&- H_{b_1 k}^b T_{\beta_1 \dots \beta_q b b_2 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} - \dots - H_{b_s k}^b T_{\beta_1 \dots \beta_q b_1 \dots b_{s-1} b}^{\alpha_1 \dots \alpha_p a_1 \dots a_r}
\end{aligned}$$

and

$$\begin{aligned}
(4.7)'' \quad T_{j_1 \dots j_q b_1 \dots b_s}^{i_1 \dots i_p a_1 \dots a_r} |_{c=} &= \dot{\partial}_c \left( T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \right) \\
&+ V_{ic}^{i_1} T_{j_1 \dots j_q b_1 \dots b_s}^{i_2 \dots i_p a_1 \dots a_r} + \dots + V_{ic}^{i_p} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{i_1 \dots i_{p-1} a_1 \dots a_r} \\
&- V_{j_1 c}^{j_1} T_{j_2 \dots j_q b_1 \dots b_s}^{i_1 \dots i_p a_1 \dots a_r} - \dots - V_{j_q c}^{j_q} T_{j_1 \dots j_{q-1} b_1 \dots b_s}^{i_1 \dots i_p a_1 \dots a_r} \\
&+ V_{a c}^{a_1} T_{j_1 \dots j_q b_1 \dots b_s}^{i_1 \dots i_p a_2 \dots a_r} + \dots + V_{a c}^{a_r} T_{j_1 \dots j_q b_1 \dots b_s}^{i_1 \dots i_p a_1 \dots a_{r-1} a} \\
&- V_{b_1 c}^b T_{j_1 \dots j_q b b_2 \dots b_s}^{i_1 \dots i_p a_1 \dots a_r} - \dots - V_{b_s c}^b T_{j_1 \dots j_q b_1 \dots b_{s-1} b}^{i_1 \dots i_p a_1 \dots a_r}.
\end{aligned}$$

**Definition 4.2** If  $(E, \pi, M) = (F, \nu, N)$ ,  $(\rho, \eta) \Gamma$  is a  $(\rho, \eta)$ -connection for the vector bundle  $(E, \pi, M)$  and

$$\left( (\rho, \eta) H_{bc}^a, (\rho, \eta) \tilde{H}_{bc}^a, (\rho, \eta) V_{bc}^a, (\rho, \eta) \tilde{V}_{bc}^a \right)$$

are the components of a distinguished linear  $(\rho, \eta)$ -connection for the generalized tangent bundle  $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$  such that

$$(\rho, \eta) H_{bc}^a = (\rho, \eta) \tilde{H}_{bc}^a \text{ and } (\rho, \eta) V_{bc}^a = (\rho, \eta) \tilde{V}_{bc}^a,$$

then we will say that *the generalized tangent bundle  $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$  is endowed with a normal distinguished linear  $(\rho, \eta)$ -connection on components  $((\rho, \eta) H_{bc}^a, (\rho, \eta) V_{bc}^a)$ .*

In the particular case of Lie algebroids,  $(\eta, h) = (Id_M, Id_M)$ , the components of a normal distinguished linear  $(\rho, Id_M)$ -connection  $(\rho H, \rho V)$  will be denoted  $(\rho H_{bc}^a, \rho V_{bc}^a)$ .

In the classical case,  $(\rho, \eta, h) = (Id_{TE}, Id_M, Id_M)$ , the components of a normal distinguished linear  $(Id_{TM}, Id_M)$ -connection  $(H, V)$  will be denoted  $(H_{jk}^i, V_{jk}^i)$ .

## 5 The $(\rho, \eta)$ -(pseudo)metrizability

We consider the following diagram:

$$\begin{array}{ccc} E & & (F, [, ]_{F,h}, (\rho, \eta)) \\ \pi \downarrow & & \downarrow \nu \\ M & \xrightarrow{h} & N \end{array}$$

where  $(E, \pi, M) \in |\mathbf{B}^V|$  and  $((F, \nu, N), [, ]_{F,h}, (\rho, \eta))$  is a generalized Lie algebroid. Let  $(\rho, \eta) \Gamma$  be a  $(\rho, \eta)$ -connection for the vector bundle  $(E, \pi, M)$  and let  $((\rho, \eta) H, (\rho, \eta) V)$  be a distinguished linear  $(\rho, \eta)$ -connection for the generalized tangent bundle

$$((\rho, \eta) TE, (\rho, \eta) \tau_E, E).$$

**Definition 5.1** A tensor  $d$ -field

$$G = g_{\alpha\beta} d\tilde{z}^\alpha \otimes d\tilde{z}^\beta + g_{ab} \delta\tilde{y}^a \otimes \delta\tilde{y}^b \in \mathcal{DT}_{22}^{00}((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$$

will be called *pseudometrical structure* if its components are symmetric and the matrices  $\|g_{\alpha\beta}(u_x)\|$  and  $\|g_{ab}(u_x)\|$  are nondegenerate, for any point  $u_x \in E$ .

Moreover, if the matrices  $\|g_{\alpha\beta}(u_x)\|$  and  $\|g_{ab}(u_x)\|$  has constant signature, then the tensor  $d$ -field  $G$  will be called *metrical structure*.

Let

$$G = g_{\alpha\beta} d\tilde{z}^\alpha \otimes d\tilde{z}^\beta + g_{ab} \delta\tilde{y}^a \otimes \delta\tilde{y}^b$$

be a (pseudo)metrical structure. If  $\alpha, \beta \in \overline{1, p}$  and  $a, b \in \overline{1, r}$ , then for any vector local  $(m+r)$ -chart  $(U, s_U)$  of  $(E, \pi, M)$ , we consider the real functions

$$\pi^{-1}(U) \xrightarrow{\tilde{g}^{\beta\alpha}} \mathbb{R}$$

and

$$\pi^{-1}(U) \xrightarrow{\tilde{g}^{ba}} \mathbb{R}$$

such that

$$\|\tilde{g}^{\beta\alpha}(u_x)\| = \|g_{\alpha\beta}(u_x)\|^{-1}$$

and

$$\|\tilde{g}^{ba}(u_x)\| = \|g_{ab}(u_x)\|^{-1},$$

for any  $u_x \in \pi^{-1}(U) \setminus \{0_x\}$ .

**Definition 5.2** If around each point  $x \in M$  it exists a local vector  $m + r$ -chart  $(U, s_U)$  and a local  $m$ -chart  $(U, \xi_U)$  such that  $g_{\alpha\beta} \circ s_U^{-1} \circ (\xi_U^{-1} \times Id_{\mathbb{R}^m})(x, y)$  and  $g_{ab} \circ s_U^{-1} \circ (\xi_U^{-1} \times Id_{\mathbb{R}^m})(x, y)$  depends only on  $x$ , for any  $u_x \in \pi^{-1}(U)$ , then we will say that the *(pseudo)metrical structure*

$$G = g_{\alpha\beta} d\tilde{z}^\alpha \otimes d\tilde{z}^\beta + g_{ab} \delta\tilde{y}^a \otimes \delta\tilde{y}^b$$

is a *Riemannian (pseudo)metrical structure*.

If only the condition is verified:

" $g_{\alpha\beta} \circ s_U^{-1} \circ (\xi_U^{-1} \times Id_{\mathbb{R}^m})(x, y)$  depends only on  $x$ , for any  $u_x \in \pi^{-1}(U)$ " respectively " $g_{ab} \circ s_U^{-1} \circ (\xi_U^{-1} \times Id_{\mathbb{R}^m})(x, y)$  depends only on  $x$ , for any  $u_x \in \pi^{-1}(U)$ ", then we will say that the *(pseudo)metrical structure  $G$  is a Riemannian  $\mathcal{H}$ -(pseudo)metrical structure* respectively a *Riemannian  $\mathcal{V}$ -(pseudo)metrical structure*.

**Definition 5.3** If around each point  $x \in M$  there exists a local vector  $m + r$ -chart  $(U, s_U)$  and a local  $m$ -chart  $(U, \xi_U)$  such that  $g_{\alpha\beta} \circ s_U^{-1} \circ (\xi_U^{-1} \times Id_{\mathbb{R}^m})(x, y)$  and  $g_{ab} \circ s_U^{-1} \circ (\xi_U^{-1} \times Id_{\mathbb{R}^m})(x, y)$  depends only on  $y$ , for any  $u_x \in \pi^{-1}(U)$ , then we will say that the *(pseudo)metrical structure*

$$G = g_{\alpha\beta} d\tilde{z}^\alpha \otimes d\tilde{z}^\beta + g_{ab} \delta\tilde{y}^a \otimes \delta\tilde{y}^b$$

is a *Minkowski (pseudo)metrical structure*.

If only the condition is verified:

" $g_{\alpha\beta} \circ s_U^{-1} \circ (\xi_U^{-1} \times Id_{\mathbb{R}^m})(x, y)$  depends only on  $y$ , for any  $u_x \in \pi^{-1}(U)$ " respectively " $g_{ab} \circ s_U^{-1} \circ (\xi_U^{-1} \times Id_{\mathbb{R}^m})(x, y)$  depends only on  $y$ , for any  $u_x \in \pi^{-1}(U)$ ", then we will say that the *(pseudo)metrical structure  $G$  is a Minkowski  $\mathcal{H}$ -(pseudo)metrical structure* respectively a *Minkowski  $\mathcal{V}$ -(pseudo)metrical structure*.

**Definition 5.4** If there exists a *(pseudo)metrical structure*

$$G = g_{\alpha\beta} d\tilde{z}^\alpha \otimes d\tilde{z}^\beta + g_{ab} \delta\tilde{y}^a \otimes \delta\tilde{y}^b$$

and a distinguished linear  $(\rho, \eta)$ -connection

$$((\rho, \eta) H, (\rho, \eta) V)$$

such that

$$(5.1) \quad (\rho, \eta) D_X G = 0, \quad \forall X \in \Gamma((\rho, \eta) TE, (\rho, \eta) \tau_E, E).$$

then the generalized tangent bundle

$$((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$$

will be called  *$(\rho, \eta)$ -(pseudo)metrizable*

Condition (5.1) is equivalent with the following equalities:

$$(5.2) \quad g_{\alpha\beta}|_\gamma = 0, \quad g_{ab}|_\gamma = 0, \quad g_{\alpha\beta}|_c = 0, \quad g_{ab}|_c = 0.$$



If  $g_{\alpha\beta}|_\gamma=0$  and  $g_{ab}|_\gamma=0$ , then we will say that the vector bundle  $((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$  is  $\mathcal{H}$ -( $\rho, \eta$ )-(pseudo)metrizable.

If  $g_{\alpha\beta}|_c=0$  and  $g_{ab}|_c=0$ , then we will say that the vector bundle  $((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$  is  $\mathcal{V}$ -( $\rho, \eta$ )-(pseudo)metrizable.

**Theorem 5.1** If  $((\rho, \eta)\mathring{H}, (\rho, \eta)\mathring{V})$  is a distinguished linear  $(\rho, \eta)$ -connection for the generalized tangent bundle  $((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$  and  $G = g_{\alpha\beta}d\tilde{z}^\alpha \otimes d\tilde{z}^\beta + g_{ab}\delta\tilde{y}^a \otimes \delta\tilde{y}^b$  is a (pseudo)metrical structure, then the following real local functions:

$$\begin{aligned}
(\rho, \eta) H_{b\gamma}^\alpha &= \frac{1}{2}\tilde{g}^{\alpha\varepsilon} \left( \Gamma(\tilde{\rho}, Id_E) \left( \tilde{\delta}_\gamma \right) g_{\varepsilon\beta} + \Gamma(\tilde{\rho}, Id_E) \left( \tilde{\delta}_\beta \right) g_{\varepsilon\gamma} - \Gamma(\tilde{\rho}, Id_E) \left( \tilde{\delta}_\varepsilon \right) g_{\beta\gamma} \right. \\
&\quad \left. + g_{\theta\varepsilon} L_{\gamma\beta}^\theta \circ h \circ \pi - g_{\beta\theta} L_{\gamma\varepsilon}^\theta \circ h \circ \pi - g_{\theta\gamma} L_{\beta\varepsilon}^\theta \circ h \circ \pi \right), \\
(\rho, \eta) H_{b\gamma}^a &= (\rho, \eta) \mathring{H}_{b\gamma}^a + \frac{1}{2}\tilde{g}^{ac} g_{bc|_\gamma}^0, \\
(\rho, \eta) V_{\beta c}^\alpha &= (\rho, \eta) \mathring{V}_{\beta c}^\alpha + \frac{1}{2}\tilde{g}^{\alpha\varepsilon} g_{\beta\varepsilon|_c}^0, \\
(\rho, \eta) V_{bc}^a &= \frac{1}{2}\tilde{g}^{ae} \left( \Gamma(\tilde{\rho}, Id_E) \left( \dot{\tilde{\delta}}_c \right) g_{eb} + \Gamma(\tilde{\rho}, Id_E) \left( \dot{\tilde{\delta}}_b \right) g_{ec} - \Gamma(\tilde{\rho}, Id_E) \left( \dot{\tilde{\delta}}_e \right) g_{bc} \right)
\end{aligned}
\tag{5.3}$$

are components of a distinguished linear  $(\rho, \eta)$ -connection such that the generalized tangent bundle  $((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$  becomes  $(\rho, \eta)$ -(pseudo)metrizable.

**Corollary 5.1** In the particular case of Lie algebroids,  $(\eta, h) = (Id_M, Id_M)$ , then we obtain

$$\begin{aligned}
\rho H_{b\gamma}^\alpha &= \frac{1}{2}\tilde{g}^{\alpha\varepsilon} \left( \Gamma(\tilde{\rho}, Id_E) \left( \tilde{\delta}_\gamma \right) g_{\varepsilon\beta} + \Gamma(\tilde{\rho}, Id_E) \left( \tilde{\delta}_\beta \right) g_{\varepsilon\gamma} - \Gamma(\tilde{\rho}, Id_E) \left( \tilde{\delta}_\varepsilon \right) g_{\beta\gamma} \right. \\
&\quad \left. + g_{\theta\varepsilon} L_{\gamma\beta}^\theta \circ \pi - g_{\beta\theta} L_{\gamma\varepsilon}^\theta \circ \pi - g_{\theta\gamma} L_{\beta\varepsilon}^\theta \circ \pi \right), \\
\rho H_{b\gamma}^a &= \rho \mathring{H}_{b\gamma}^a + \frac{1}{2}\tilde{g}^{ac} g_{bc|_\gamma}^0, \\
\rho V_{\beta c}^\alpha &= \rho \mathring{V}_{\beta c}^\alpha + \frac{1}{2}\tilde{g}^{\alpha\varepsilon} g_{\beta\varepsilon|_c}^0, \\
\rho V_{bc}^a &= \frac{1}{2}\tilde{g}^{ae} \left( \Gamma(\tilde{\rho}, Id_E) \left( \dot{\tilde{\delta}}_c \right) g_{eb} + \Gamma(\tilde{\rho}, Id_E) \left( \dot{\tilde{\delta}}_b \right) g_{ec} - \Gamma(\tilde{\rho}, Id_E) \left( \dot{\tilde{\delta}}_e \right) g_{bc} \right)
\end{aligned}
\tag{5.3}'$$

In the classical case,  $(\rho, \eta, h) = (Id_{TE}, Id_M, Id_M)$ , then we obtain

$$\begin{aligned}
H_{jk}^i &= \frac{1}{2}\tilde{g}^{ih} (\delta_k g_{hj} + \delta_j g_{hk} - \delta_h g_{jk}) \\
H_{bk}^a &= \mathring{H}_{bk}^a + \frac{1}{2}\tilde{g}^{ac} g_{bc|_k}^0, \\
V_{jc}^i &= \mathring{V}_{jc}^i + \frac{1}{2}\tilde{g}^{ih} g_{jh|_c}^0, \\
V_{bc}^a &= \frac{1}{2}\tilde{g}^{ae} \left( \dot{\partial}_c g_{eb} + \dot{\partial}_b g_{ec} - \dot{\partial}_e g_{bc} \right)
\end{aligned}
\tag{5.3}''$$

**Theorem 5.2** If the distinguished linear  $(\rho, \eta)$ -connection  $((\rho, \eta)\mathring{H}, (\rho, \eta)\mathring{V})$  coincides with the Berwald linear  $(\rho, \eta)$ -connection in the previous theorem, then the local real

functions:

$$\begin{aligned}
(\rho, \eta) \overset{c}{H}_{\beta\gamma}^{\alpha} &= \frac{1}{2} \tilde{g}^{\alpha\epsilon} \left( \Gamma(\tilde{\rho}, Id_E) \left( \tilde{\delta}_{\gamma} \right) g_{\epsilon\beta} + \Gamma(\tilde{\rho}, Id_E) \left( \tilde{\delta}_{\beta} \right) g_{\epsilon\gamma} \right. \\
&\quad \left. - \Gamma(\tilde{\rho}, Id_E) \left( \tilde{\delta}_{\epsilon} \right) g_{\beta\gamma} + g_{\theta\epsilon} L_{\gamma\beta}^{\theta} \circ h \circ \pi - g_{\beta\theta} L_{\gamma\epsilon}^{\theta} \circ h \circ \pi - g_{\theta\gamma} L_{\beta\epsilon}^{\theta} \circ h \circ \pi \right), \\
(5.4) \quad (\rho, \eta) \overset{c}{H}_{b\gamma}^a &= \frac{\partial(\rho, \eta) \Gamma_{\gamma}^a}{\partial y^b} + \frac{1}{2} \tilde{g}^{ac} g_{bc|_{\gamma}}^0, \\
(\rho, \eta) \overset{c}{V}_{\beta c}^{\alpha} &= \frac{1}{2} \tilde{g}^{\alpha\epsilon} \frac{\partial g_{\beta\epsilon}}{\partial y^c}, \\
(\rho, \eta) \overset{c}{V}_{bc}^a &= \frac{1}{2} \tilde{g}^{ae} \left( \frac{\partial g_{e\beta}}{\partial y^c} + \frac{\partial g_{ec}}{\partial y^b} - \frac{\partial g_{bc}}{\partial y^e} \right)
\end{aligned}$$

are the components of a distinguished linear  $(\rho, \eta)$ -connection such that the generalized tangent bundle  $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$  becomes  $(\rho, \eta)$ -(pseudo)metrizable.

Moreover, if the (pseudo)metrical structure  $G$  is  $\mathcal{H}$ - and  $\mathcal{V}$ -Riemannian, then the local real functions:

$$\begin{aligned}
(\rho, \eta) \overset{c}{H}_{\beta\gamma}^{\alpha} &= \frac{1}{2} \tilde{g}^{\alpha\epsilon} \left( \rho_{\gamma}^k \circ h \circ \pi \frac{\partial g_{\epsilon\beta}}{\partial x^k} + \rho_{\beta}^j \circ h \circ \pi \frac{\partial g_{\epsilon\gamma}}{\partial x^j} - \rho_{\epsilon}^e \circ h \circ \pi \frac{\partial g_{\beta\gamma}}{\partial x^e} + \right. \\
&\quad \left. + g_{\theta\epsilon} L_{\gamma\beta}^{\theta} \circ h \circ \pi - g_{\beta\theta} L_{\gamma\epsilon}^{\theta} \circ h \circ \pi - g_{\theta\gamma} L_{\beta\epsilon}^{\theta} \circ h \circ \pi \right), \\
(5.5) \quad (\rho, \eta) \overset{c}{H}_{b\gamma}^a &= \frac{\partial(\rho, \eta) \Gamma_{\gamma}^a}{\partial y^b} + \frac{1}{2} \tilde{g}^{ac} \left( \rho_{\gamma}^i \circ h \circ \pi \frac{\partial g_{bc}}{\partial x^i} - \frac{\partial(\rho, \eta) \Gamma_{\gamma}^e}{\partial y^b} g_{ec} - \frac{\partial(\rho, \eta) \Gamma_{\gamma}^e}{\partial y^c} g_{eb} \right), \\
(\rho, \eta) \overset{c}{V}_{\beta c}^{\alpha} &= 0, \\
(\rho, \eta) \overset{c}{V}_{bc}^a &= 0.
\end{aligned}$$

are the components of a distinguished linear  $(\rho, \eta)$ -connection such that the generalized tangent bundle  $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$  becomes  $(\rho, \eta)$ -(pseudo)metrizable.

**Corollary 5.2** In the particular case of Lie algebroids,  $(\eta, h) = (Id_M, Id_M)$ , then we obtain

$$\begin{aligned}
\rho \overset{c}{H}_{\beta\gamma}^{\alpha} &= \frac{1}{2} \tilde{g}^{\alpha\epsilon} \left( \Gamma(\tilde{\rho}, Id_E) \left( \tilde{\delta}_{\gamma} \right) g_{\epsilon\beta} + \Gamma(\tilde{\rho}, Id_E) \left( \tilde{\delta}_{\beta} \right) g_{\epsilon\gamma} \right. \\
&\quad \left. - \Gamma(\tilde{\rho}, Id_E) \left( \tilde{\delta}_{\epsilon} \right) g_{\beta\gamma} + g_{\theta\epsilon} L_{\gamma\beta}^{\theta} \circ \pi - g_{\beta\theta} L_{\gamma\epsilon}^{\theta} \circ \pi - g_{\theta\gamma} L_{\beta\epsilon}^{\theta} \circ \pi \right) \\
(5.4)' \quad \rho \overset{c}{H}_{b\gamma}^a &= \frac{\partial \rho \Gamma_{\gamma}^a}{\partial y^b} + \frac{1}{2} \tilde{g}^{ac} g_{bc|_{\gamma}}^0, \\
\rho \overset{c}{V}_{\beta c}^{\alpha} &= \frac{1}{2} \tilde{g}^{\alpha\epsilon} \frac{\partial g_{\beta\epsilon}}{\partial y^c}, \\
\rho \overset{c}{V}_{bc}^a &= \frac{1}{2} \tilde{g}^{ae} \left( \frac{\partial g_{e\beta}}{\partial y^c} + \frac{\partial g_{ec}}{\partial y^b} - \frac{\partial g_{bc}}{\partial y^e} \right)
\end{aligned}$$

If the (pseudo)metrical structure  $G$  is  $\mathcal{H}$ - and  $\mathcal{V}$ -Riemannian, then

$$\begin{aligned}
\rho \overset{c}{H}_{\beta\gamma}^{\alpha} &= \frac{1}{2} \tilde{g}^{\alpha\epsilon} \left( \rho_{\gamma}^k \circ \pi \frac{\partial g_{\epsilon\beta}}{\partial x^k} + \rho_{\beta}^j \circ \pi \frac{\partial g_{\epsilon\gamma}}{\partial x^j} - \rho_{\epsilon}^e \circ \pi \frac{\partial g_{\beta\gamma}}{\partial x^e} + \right. \\
&\quad \left. + g_{\theta\epsilon} L_{\gamma\beta}^{\theta} \circ \pi - g_{\beta\theta} L_{\gamma\epsilon}^{\theta} \circ \pi - g_{\theta\gamma} L_{\beta\epsilon}^{\theta} \circ \pi \right), \\
(5.5)' \quad \rho \overset{c}{H}_{b\gamma}^a &= \frac{\partial \rho \Gamma_{\gamma}^a}{\partial y^b} + \frac{1}{2} \tilde{g}^{ac} \left( \rho_{\gamma}^i \circ \pi \frac{\partial g_{bc}}{\partial x^i} - \frac{\partial \rho \Gamma_{\gamma}^e}{\partial y^b} g_{ec} - \frac{\partial \rho \Gamma_{\gamma}^e}{\partial y^c} g_{eb} \right), \\
\rho \overset{c}{V}_{\beta c}^{\alpha} &= 0, \quad \rho \overset{c}{V}_{bc}^a = 0
\end{aligned}$$

In the classicale case,  $(\rho, \eta, h) = (Id_{TE}, Id_M, Id_M)$ , then we obtain

$$\begin{aligned}
\overset{c}{H}_{jk}^i &= \frac{1}{2} \tilde{g}^{ih} (\delta_k g_{hj} + \delta_j g_{hk} - \delta_h g_{jk}) \\
\overset{c}{H}_{bk}^a &= \frac{\partial \Gamma_k^a}{\partial y^b} + \frac{1}{2} \tilde{g}^{ac} g_{bc|k}^{\circ}, \\
\overset{c}{V}_{jc}^i &= \frac{1}{2} \tilde{g}^{ih} \frac{\partial g_{jh}}{\partial y^c}, \\
\overset{c}{V}_{bc}^a &= \frac{1}{2} \tilde{g}^{ae} \left( \frac{\partial g_{e\beta}}{\partial y^c} + \frac{\partial g_{ec}}{\partial y^b} - \frac{\partial g_{bc}}{\partial y^e} \right)
\end{aligned}
\tag{5.4}''$$

If the (pseudo)metrical structure  $G$  is  $\mathcal{H}$ - and  $\mathcal{V}$ -Riemannian, then

$$\begin{aligned}
\overset{c}{H}_{jk}^i &= \frac{1}{2} \tilde{g}^{ih} \left( \frac{\partial g_{hj}}{\partial x^k} + \frac{\partial g_{hk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^h} \right) \\
\overset{c}{H}_{bk}^a &= \frac{\partial \Gamma_k^a}{\partial y^b} + \frac{1}{2} \tilde{g}^{ac} \left( \frac{\partial g_{bc}}{\partial x^i} - \frac{\partial \Gamma_k^e}{\partial y^b} g_{ec} - \frac{\partial \Gamma_k^e}{\partial y^c} g_{eb} \right), \\
\overset{c}{V}_{jc}^i &= 0, \quad \overset{c}{V}_{bc}^a = 0
\end{aligned}
\tag{5.5}''$$

**Theorem 5.3** Let  $(\rho, \eta) \Gamma$  be a  $(\rho, \eta)$ -connection for the vector bundle  $(E, \pi, M)$ . Let

$$((\rho, \eta) \overset{\circ}{H}, (\rho, \eta) \overset{\circ}{V})$$

be a distinguished linear  $(\rho, \eta)$ -connection for  $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$  and let

$$G = g_{\alpha\beta} d\tilde{z}^\alpha \otimes d\tilde{z}^\beta + g_{ab} \delta\tilde{y}^a \otimes \delta\tilde{y}^b$$

be a (pseudo)metrical structure.

Let

$$\begin{aligned}
O_{\beta\gamma}^{\alpha\varepsilon} &= \frac{1}{2} (\delta_\beta^\alpha \delta_\gamma^\varepsilon - g_{\beta\gamma} \tilde{g}^{\alpha\varepsilon}), \quad O_{\beta\gamma}^{*\alpha\varepsilon} = \frac{1}{2} (\delta_\beta^\alpha \delta_\gamma^\varepsilon + g_{\beta\gamma} \tilde{g}^{\alpha\varepsilon}), \\
O_{bc}^{ae} &= \frac{1}{2} (\delta_b^a \delta_c^e - g_{bc} \tilde{g}^{ae}), \quad O_{bc}^{*ae} = \frac{1}{2} (\delta_b^a \delta_c^e + g_{bc} \tilde{g}^{ae}),
\end{aligned}
\tag{5.6}$$

be the Obata operators.

If the real local functions  $X_{\beta\gamma}^\alpha, X_{\beta c}^\alpha, Y_{b\gamma}^a, Y_{bc}^a$  are components of tensor fields, then the local real functions given in the following:

$$\begin{aligned}
(\rho, \eta) H_{\beta\gamma}^\alpha &= (\rho, \eta) \overset{c}{H}_{\beta\gamma}^\alpha + O_{\gamma\eta}^{\alpha\varepsilon} X_{\varepsilon\beta}^\eta, \\
(\rho, \eta) H_{b\gamma}^a &= (\rho, \eta) \overset{c}{H}_{b\gamma}^a + O_{bd}^{ae} Y_{e\gamma}^d, \\
(\rho, \eta) V_{\beta c}^\alpha &= (\rho, \eta) \overset{c}{V}_{\beta c}^\alpha + O_{\beta\eta}^{*\alpha\varepsilon} X_{\varepsilon c}^\eta, \\
(\rho, \eta) V_{bc}^a &= (\rho, \eta) \overset{c}{V}_{bc}^a + O_{bd}^{*ae} Y_{ec}^d,
\end{aligned}
\tag{5.7}$$

are the components of a distinguished linear  $(\rho, \eta)$ -connection such that the generalized tangent bundle  $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$  becomes  $(\rho, \eta)$ -(pseudo)metrizable.

**Corollary 5.3** *In the particular case of Lie algebroids,  $(\eta, h) = (Id_M, Id_M)$ , then we obtain*

$$\begin{aligned}
 \rho H_{\beta\gamma}^\alpha &= \rho H_{\beta\gamma}^{\alpha c} + O_{\gamma\eta}^{\alpha\varepsilon} X_{\varepsilon\beta}^\eta, \\
 \rho H_{b\gamma}^a &= \rho H_{b\gamma}^{ac} + O_{bd}^{ae} Y_{e\gamma}^d, \\
 \rho V_{\beta c}^\alpha &= \rho V_{\beta c}^{\alpha c} + O_{\beta\eta}^{*\alpha\varepsilon} X_{\varepsilon c}^\eta, \\
 \rho V_{bc}^a &= \rho V_{bc}^{ac} + O_{bd}^{*ae} Y_{ec}^d,
 \end{aligned}
 \tag{5.7}'$$

*In the classicale case,  $(\rho, \eta, h) = (Id_{TE}, Id_M, Id_M)$ , then we obtain (see [17])*

$$\begin{aligned}
 H_{jk}^i &= H_{jk}^{ic} + O_{kh}^{il} X_{lj}^h, \\
 H_{bk}^a &= H_{bk}^{ac} + O_{bd}^{ae} Y_{ek}^d, \\
 V_{jc}^i &= V_{jc}^{ic} + O_{jh}^{*il} X_{lc}^h, \\
 \rho V_{bc}^a &= V_{bc}^{ac} + O_{bd}^{*ae} Y_{ec}^d,
 \end{aligned}
 \tag{5.7}''$$

**Theorem 5.4** *Let  $(\rho, \eta) \Gamma$  be a  $(\rho, \eta)$ -connection for the vector bundle  $(E, \pi, M)$ .*

*If*

$$((\rho, \eta) \mathring{H}, (\rho, \eta) \mathring{V})$$

*is a distinguished linear  $(\rho, \eta)$ -connection for the generalized tangent bundle*

$$((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$$

*and*

$$G = g_{\alpha\beta} d\tilde{z}^\alpha \otimes d\tilde{z}^\beta + g_{ab} \delta\tilde{y}^a \otimes \delta\tilde{y}^b$$

*is a (pseudo)metrical structure, then the real local functions:*

$$\begin{aligned}
 (\rho, \eta) H_{\beta\gamma}^\alpha &= (\rho, \eta) \mathring{H}_{\beta\gamma}^\alpha + \frac{1}{2} \tilde{g}^{\alpha\varepsilon} g_{\varepsilon\beta|_\gamma}^0, \\
 (\rho, \eta) H_{b\gamma}^a &= (\rho, \eta) \mathring{H}_{b\gamma}^a + \frac{1}{2} \tilde{g}^{ae} g_{eb|_\gamma}^0, \\
 (\rho, \eta) V_{\beta c}^\alpha &= (\rho, \eta) \mathring{V}_{\beta c}^\alpha + \frac{1}{2} \tilde{g}^{\alpha\varepsilon} g_{\varepsilon\beta}^0|_c, \\
 (\rho, \eta) V_{bc}^a &= (\rho, \eta) \mathring{V}_{bc}^a + \frac{1}{2} \tilde{g}^{ae} g_{eb}^0|_c
 \end{aligned}
 \tag{5.8}$$

*are the components of a distinguished linear  $(\rho, \eta)$ -connection such that the generalized tangent bundle  $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$  becomes  $(\rho, \eta)$ -(pseudo)metrizable.*

**Corollary 5.4** *In the particular case of Lie algebroids,  $(\eta, h) = (Id_M, Id_M)$ , then*

we obtain

$$\begin{aligned}
(5.8)' \quad \rho H_{\beta\gamma}^\alpha &= \rho \mathring{H}_{\beta\gamma}^\alpha + \frac{1}{2} \tilde{g}^{\alpha\varepsilon} g_{\varepsilon\beta|_\gamma}^0, \\
\rho H_{b\gamma}^a &= \rho \mathring{H}_{b\gamma}^a + \frac{1}{2} \tilde{g}^{ae} g_{eb|_\gamma}^0, \\
\rho V_{\beta c}^\alpha &= \rho \mathring{V}_{\beta c}^\alpha + \frac{1}{2} \tilde{g}^{\alpha\varepsilon} g_{\varepsilon\beta}^0|_c, \\
\rho V_{bc}^a &= \rho \mathring{V}_{bc}^a + \frac{1}{2} \tilde{g}^{ae} g_{eb}^0|_c
\end{aligned}$$

In the classical case,  $(\rho, \eta, h) = (Id_{TE}, Id_M, Id_M)$ , then we obtain (see [15])

$$\begin{aligned}
(5.8)'' \quad H_{jk}^i &= \mathring{H}_{jk}^i + \frac{1}{2} \tilde{g}^{ih} g_{hj|k}^0, \\
H_{bk}^a &= \mathring{H}_{bk}^a + \frac{1}{2} \tilde{g}^{ae} g_{eb|k}^0, \\
V_{jc}^i &= \mathring{V}_{jc}^i + \frac{1}{2} \tilde{g}^{ih} g_{hj}^0|_c, \\
V_{bc}^a &= \mathring{V}_{bc}^a + \frac{1}{2} \tilde{g}^{ae} g_{eb}^0|_c
\end{aligned}$$

## 6 Generalized Lagrange $(\rho, \eta)$ -spaces, Lagrange $(\rho, \eta)$ -spaces and Finsler $(\rho, \eta)$ -spaces

We consider the following diagram:

$$\begin{array}{ccc}
E & & (F, [, ]_{F,h}, (\rho, \eta)) \\
\pi \downarrow & & \downarrow \nu \\
M & \xrightarrow{h} & N
\end{array}$$

such that  $(E, \pi, M) = (F, \nu, N)$  and the generalized tangent bundle

$$((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$$

is  $(\rho, \eta)$ -(pseudo)metrizable. Let

$$G = h_{ab} dz^a \otimes dz^a + g_{ab} \delta \tilde{y}^a \otimes \delta \tilde{y}^b$$

be a (pseudo)metrical structure and let

$$((\rho, \eta) H, (\rho, \eta) V)$$

be a distinguished linear  $(\rho, \eta)$ -connection such that

$$(\rho, \eta) D_X G = 0, \quad \forall X \in \Gamma((\rho, \eta) TE, (\rho, \eta) \tau_E, E).$$

**Definition 6.1** A smooth *Lagrange fundamental function* on the vector bundle  $(E, \pi, M)$  is a mapping  $E \xrightarrow{L} \mathbb{R}$  which satisfies the following conditions:

1.  $L \circ u \in C^\infty(M)$ , for any  $u \in \Gamma(E, \pi, M) \setminus \{0\}$ ;
2.  $L \circ 0 \in C^0(M)$ , where 0 means the null section of  $(E, \pi, M)$ .

If  $(U, s_U)$  is a local vector  $(m+r)$ -chart for  $(E, \pi, M)$ , then the real function

$$L_{ab} \stackrel{put}{=} \frac{\partial^2 L}{\partial y^a \partial y^b} \stackrel{put}{=} \frac{\partial}{\partial y^a} \left( \frac{\partial}{\partial y^b} (L) \right)$$

is defined on  $\pi^{-1}(U)$ .

**Definition 6.2** If for any local vector  $m+r$ -chart  $(U, s_U)$  of  $(E, \pi, M)$ , we have:

$$(6.2) \quad \text{rank} \|L_{ab}(u_x)\| = r,$$

for any  $u_x \in \pi^{-1}(U) \setminus \{0_x\}$ , then we will say that the Lagrangian  $L$  is regular.

**Proposition 6.1** If the Lagrangian  $L$  is regular, then for any local vector  $m+r$ -chart  $(U, s_U)$  of  $(E, \pi, M)$ , we obtain the real functions  $\tilde{L}^{ba}$  locally defined by

$$(6.3) \quad \begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\tilde{L}^{ba}} & \mathbb{R} \\ u_x & \longmapsto & \tilde{L}^{ba}(u_x) \end{array},$$

where  $\|\tilde{L}^{ba}(u_x)\| = \|L_{ab}(u_x)\|^{-1}$ , for any  $u_x \in \pi^{-1}(U) \setminus \{0_x\}$ .

**Definition 6.3** A smooth Finsler fundamental function on the vector bundle  $(E, \pi, M)$  is a smooth Lagrange fundamental function  $E \xrightarrow{F} \mathbb{R}_+$  which satisfies the following conditions:

1.  $F$  is positively 1-homogenous on the fibres of vector bundle  $(E, \pi, M)$ ;
2. For any local vector  $m+r$ -chart  $(U, s_U)$  of  $(E, \pi, M)$ , the hessian:

$$(6.4) \quad \|F_{ab}^2(u_x)\|$$

is positively define for any  $u_x \in \pi^{-1}(U) \setminus \{0_x\}$ .

**Definition 6.4** If the (pseudo)metrical structure  $G$  is determined by a (pseudo)metrical structure

$$g = g_{ab} d\tilde{y}^a \otimes d\tilde{y}^b \in \mathcal{T}_2^0(V(\rho, \eta)TE, (\rho, \eta), \tau_E, E),$$

namely

$$G = g_{ab} d\tilde{z}^a \otimes d\tilde{z}^a + g_{ab} \delta\tilde{y}^a \otimes \delta\tilde{y}^b,$$

then the  $(\rho, \eta)$ -(pseudo)metrizable vector bundle

$$((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$$

will be called the *generalized Lagrange  $(\rho, \eta)$ -space*.

In particular, if the (pseudo)metrical structure  $g$  is determined with the help of a regular Lagrange (Finsler) fundamental function, namely  $g = L_{ab} d\tilde{y}^a \otimes d\tilde{y}^b$  ( $g = F_{ab}^2 d\tilde{y}^a \otimes d\tilde{y}^b$ ), then the  $(\rho, \eta)$ -(pseudo)metrizable vector bundle

$$((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$$

will be called the *Lagrange (Finsler)  $(\rho, \eta)$ -space*.

The generalized Lagrange  $(Id_{TM}, Id_M)$ -spaces, the Lagrange  $(Id_{TM}, Id_M)$ -spaces, and the Finsler  $(Id_{TM}, Id_M)$ -spaces are the usual generalized Lagrange spaces, Lagrange spaces and Finsler spaces.

**Theorem 6.1** *If the (pseudo)metrical structure  $G$  is determined by a (pseudo)metrical structure*

$$g = g_{ab} d\tilde{y}^a \otimes d\tilde{y}^b \in \mathcal{T}_2^0(V(\rho, \eta)TE, (\rho, \eta), \tau_E, E),$$

*then, the real local functions:*

$$(6.5) \quad \begin{aligned} (\rho, \eta) H_{bc}^a &= \frac{1}{2} \tilde{g}^{ae} \left( \Gamma(\tilde{\rho}, Id_E) \left( \tilde{\delta}_b \right) g_{ec} + \Gamma(\tilde{\rho}, Id_E) \left( \tilde{\delta}_c \right) g_{be} - \Gamma(\tilde{\rho}, Id_E) \left( \tilde{\delta}_e \right) g_{bc} \right. \\ &\quad \left. - g_{cd} L_{be}^d \circ h \circ \pi + g_{bd} L_{ec}^d \circ h \circ \pi - g_{ed} L_{bc}^d \circ h \circ \pi \right), \\ (\rho, \eta) V_{bc}^a &= \frac{1}{2} \tilde{g}^{ae} \left( \Gamma(\tilde{\rho}, Id_E) \left( \dot{\tilde{\partial}}_c \right) g_{eb} + \Gamma(\tilde{\rho}, Id_E) \left( \dot{\tilde{\partial}}_b \right) g_{ec} - \Gamma(\tilde{\rho}, Id_E) \left( \dot{\tilde{\partial}}_e \right) g_{bc} \right) \end{aligned}$$

*are the components of a normal distinguished linear  $(\rho, \eta)$ -connection with  $(\rho, \eta)$ - $\mathcal{H}$  ( $\mathcal{H}\mathcal{H}$ ) and  $(\rho, \eta)$ - $\mathcal{V}$  ( $\mathcal{V}\mathcal{V}$ ) torsions free such that the generalized tangent bundle  $((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$  becomes generalized Lagrange  $(\rho, \eta)$ -space.*

This normal distinguished linear  $(\rho, \eta)$ -connection will be called *generalized linear  $(\rho, \eta)$ -connection of Levi-Civita type*.

**Corolary 6.1** *In the particular case of Lie algebroids,  $(\eta, h) = (Id_M, Id_M)$ , then we obtain*

$$(6.5)' \quad \begin{aligned} \rho H_{bc}^a &= \frac{1}{2} \tilde{g}^{ae} \left( \Gamma(\tilde{\rho}, Id_E) \left( \tilde{\delta}_b \right) g_{ec} + \Gamma(\tilde{\rho}, Id_E) \left( \tilde{\delta}_c \right) g_{be} - \Gamma(\tilde{\rho}, Id_E) \left( \tilde{\delta}_e \right) g_{bc} \right. \\ &\quad \left. - g_{cd} L_{be}^d \circ \pi + g_{bd} L_{ec}^d \circ \pi - g_{ed} L_{bc}^d \circ \pi \right), \\ \rho V_{bc}^a &= \frac{1}{2} \tilde{g}^{ae} \left( \Gamma(\tilde{\rho}, Id_E) \left( \dot{\tilde{\partial}}_c \right) g_{eb} + \Gamma(\tilde{\rho}, Id_E) \left( \dot{\tilde{\partial}}_b \right) g_{ec} - \Gamma(\tilde{\rho}, Id_E) \left( \dot{\tilde{\partial}}_e \right) g_{bc} \right) \end{aligned}$$

*In the classicale case,  $(\rho, \eta, h) = (Id_{TE}, Id_M, Id_M)$ , then we obtain*

$$(6.5)'' \quad \begin{aligned} H_{bc}^a &= \frac{1}{2} \tilde{g}^{ae} (\delta_b g_{ec} + \delta_c g_{be} - \delta_e g_{bc}) \\ V_{bc}^a &= \frac{1}{2} \tilde{g}^{ae} (\dot{\partial}_c g_{eb} + \dot{\partial}_b g_{ec} - \dot{\partial}_e g_{bc}) \end{aligned}$$

*Moreover, if  $(E, \pi, M) = (TM, \tau_M, M)$ , then we obtain*

$$(6.5)''' \quad \begin{aligned} H_{jk}^i &= \frac{1}{2} \tilde{g}^{ih} (\delta_j g_{hk} + \delta_k g_{jh} - \delta_h g_{jk}) \\ V_{jk}^i &= \frac{1}{2} \tilde{g}^{ih} (\dot{\partial}_k g_{hj} + \dot{\partial}_j g_{hk} - \dot{\partial}_h g_{jk}) \end{aligned}$$

**Theorem 6.2** *Let  $((\rho, \eta)H, (\rho, \eta)V)$  be the normal distinguished linear  $(\rho, \eta)$ -connection presented in the previous theorem.*

*If*

$$\mathbb{T}_{bc}^a \tilde{\delta}_a \otimes d\tilde{z}^b \otimes d\tilde{z}^c \in \mathcal{T}_{20}^{10}((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$$

*and*

$$\mathbb{S}_{bc}^a \dot{\tilde{\partial}}_a \otimes \delta \tilde{y}^b \otimes \delta \tilde{y}^c \in \mathcal{T}_{02}^{01}((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$$

*such that they satisfy the conditions:*

$$\mathbb{T}_{bc}^a = -\mathbb{T}_{cb}^a \wedge \mathbb{S}_{bc}^a = -\mathbb{S}_{cb}^a, \quad \forall b, c \in \overline{1, n},$$

then the following real local functions:

$$(6.6) \quad \begin{aligned} (\rho, \eta) \tilde{H}_{bc}^a &= (\rho, \eta) H_{bc}^a + \frac{1}{2} \tilde{g}^{ae} \left( g_{ed} \mathbb{T}_{bc}^d - g_{bd} \mathbb{T}_{ec}^d + g_{cd} \mathbb{T}_{be}^d \right), \\ (\rho, \eta) \tilde{V}_{bc}^a &= (\rho, \eta) V_{bc}^a + \frac{1}{2} \tilde{g}^{ae} \left( g_{ed} \mathbb{S}_{bc}^d - g_{bd} \mathbb{S}_{ec}^d + g_{cd} \mathbb{S}_{be}^d \right) \end{aligned}$$

are the components of a normal distinguished linear  $(\rho, \eta)$ -connection with  $(\rho, \eta)$ - $\mathcal{H}$  ( $\mathcal{H}\mathcal{H}$ ) and  $(\rho, \eta)$ - $\mathcal{V}$  ( $\mathcal{V}\mathcal{V}$ ) torsions a priori given such that the generalized tangent bundle  $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$  derives generalized Lagrange  $(\rho, \eta)$ -space.

Moreover, we obtain:

$$(6.7) \quad \begin{aligned} \mathbb{T}_{bc}^a &= (\rho, \eta) \tilde{H}_{bc}^a - (\rho, \eta) \tilde{H}_{cb}^a - L_{bc}^a \circ h \circ \pi, \\ \mathbb{S}_{bc}^a &= (\rho, \eta) \tilde{V}_{bc}^a - (\rho, \eta) \tilde{V}_{cb}^a. \end{aligned}$$

**Corollary 6.2** In the particular case of Lie algebroids,  $(\eta, h) = (Id_M, Id_M)$ , then we obtain

$$(6.6)' \quad \begin{aligned} \rho \tilde{H}_{bc}^a &= \rho H_{bc}^a + \frac{1}{2} \tilde{g}^{ae} \left( g_{ed} \mathbb{T}_{bc}^d - g_{bd} \mathbb{T}_{ec}^d + g_{cd} \mathbb{T}_{be}^d \right), \\ \rho \tilde{V}_{bc}^a &= \rho V_{bc}^a + \frac{1}{2} \tilde{g}^{ae} \left( g_{ed} \mathbb{S}_{bc}^d - g_{bd} \mathbb{S}_{ec}^d + g_{cd} \mathbb{S}_{be}^d \right) \end{aligned}$$

and

$$(6.7)' \quad \begin{aligned} \mathbb{T}_{bc}^a &= \rho \tilde{H}_{bc}^a - \rho \tilde{H}_{cb}^a - L_{bc}^a \circ \pi, \\ \mathbb{S}_{bc}^a &= \rho \tilde{V}_{bc}^a - \rho \tilde{V}_{cb}^a. \end{aligned}$$

In the classicale case,  $(\rho, \eta, h) = (Id_{TE}, Id_M, Id_M)$ , then we obtain

$$(6.6)'' \quad \begin{aligned} \tilde{H}_{bc}^a &= H_{bc}^a + \frac{1}{2} \tilde{g}^{ae} \left( g_{ed} \mathbb{T}_{bc}^d - g_{bd} \mathbb{T}_{ec}^d + g_{cd} \mathbb{T}_{be}^d \right), \\ \tilde{V}_{bc}^a &= V_{bc}^a + \frac{1}{2} \tilde{g}^{ae} \left( g_{ed} \mathbb{S}_{bc}^d - g_{bd} \mathbb{S}_{ec}^d + g_{cd} \mathbb{S}_{be}^d \right) \end{aligned}$$

and

$$(6.7)'' \quad \begin{aligned} \mathbb{T}_{bc}^a &= \tilde{H}_{bc}^a - \tilde{H}_{cb}^a, \\ \mathbb{S}_{bc}^a &= \tilde{V}_{bc}^a - \tilde{V}_{cb}^a. \end{aligned}$$

In particular, if  $(E, \pi, M) = (TM, \tau_M, M)$ , then we obtain

$$(6.6)''' \quad \begin{aligned} \tilde{H}_{jk}^i &= H_{jk}^i + \frac{1}{2} \tilde{g}^{ie} \left( g_{eh} \mathbb{T}_{jk}^h - g_{jh} \mathbb{T}_{ek}^h + g_{kh} \mathbb{T}_{je}^h \right), \\ \tilde{V}_{jk}^i &= V_{jk}^i + \frac{1}{2} \tilde{g}^{ie} \left( g_{eh} \mathbb{S}_{jk}^h - g_{jh} \mathbb{S}_{ek}^h + g_{kh} \mathbb{S}_{je}^h \right) \end{aligned}$$

and

$$(6.7)''' \quad \begin{aligned} \mathbb{T}_{jk}^i &= \tilde{H}_{jk}^i - \tilde{H}_{kj}^i, \\ \mathbb{S}_{jk}^i &= \tilde{V}_{jk}^i - \tilde{V}_{kj}^i. \end{aligned}$$



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SECONDARY SCHOOL "CORNELIUS RADU",  
 RADINESTI VILLAGE, 217196, GORJ COUNTY, ROMANIA  
 e-mail: c\_arcus@yahoo.com, c\_arcus@radinesti.ro